Corrigendum II: Holomorphic flow of the Riemann zeta function

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Version: 27th July 2007

The proof of Theorem 4.6 of [2] is in error and we are very grateful to an anonymous referee of [3] for pointing this out. Based on the evidence of the first 500 zeros of the Riemann zeta function, the authors believe the statement of the theorem is still true, but at this time a valid proof is not available. Correct expressions for the limiting values of flow polar coordinates are given in Proposition 0.1. The best available criteria, for which the zeta function flow might have a center or node at a simple zero on the critical line, are then given in Proposition 0.2. These propositions should replace Theorem 4.6 in the original paper.

First some definitions. The Riemann-Siegel function is $Z(t) = e^{i\vartheta(t)}\zeta(\frac{1}{2}+it)$, so $Z(t)$ is real when t is real, and its zeros in $\mathbb{R}$ correspond to the zeros of $\zeta(s)$ on $\sigma = \Re z = \frac{1}{2}$, the critical line. Define the Gram points $(\frac{1}{2}+ig_n)$ to be those which satisfy $\vartheta(g_n) = n\pi$ for integral n, and anti Gram points $(\frac{1}{2}+ia_n)$ as those which satisfy $\vartheta(a_n) = (n+\frac{1}{2})\pi$. Then at Gram points $\zeta(\frac{1}{2}+ig_n)$ is real, and at anti Gram points $\zeta(\frac{1}{2}+ia_n)$ is pure imaginary. Define the function $\Lambda(z) := 2(2\pi)^{-z}\Gamma(z)\cos(\frac{\pi z}{2})$. Finally, we use the notation $\langle x\rangle := \log x$ for $x > 0$.

Proposition 0.1. *Let $z = \frac{1}{2}+i\gamma+re^{i\theta}$ define polar coordinates for a point in the neighborhood of a zero $z = \frac{1}{2}+i\gamma$ of $\zeta(z)$ on the critical line. Then for the flow $\dot{z} = \zeta(z)$ we have the formulas:*

$$\lim_{r\to 0+}\frac{\dot{r}}{r} = \frac{1}{3-2\sqrt{2}\cos(\langle 2\rangle\gamma)}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\langle n\rangle}{\sqrt{n}}[\sqrt{2}\cos(\gamma\langle\frac{n}{2}\rangle)-\cos(\gamma\langle n\rangle)],$$

$$\lim_{r\to 0+}\dot{\theta} = \frac{1}{3-2\sqrt{2}\cos(\langle 2\rangle\gamma)}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\langle n\rangle}{\sqrt{n}}[-\sqrt{2}\sin(\gamma\langle\frac{n}{2}\rangle)+\sin(\gamma\langle n\rangle)].$$

Proof. Let

$$\begin{aligned}
\delta(z) &:= \frac{1}{1-2^{1-z}},\\
\eta(z) &:= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^z}, \text{ so}\\
\zeta(z) &= \delta(z)\times\eta(z)\\
&= \frac{1}{1-2^{\frac{1}{2}-i\gamma-re^{i\theta}}}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^{\frac{1}{2}+i\gamma+re^{i\theta}}}.
\end{aligned}$$

This latter sum converges for $0 \le r < \frac{1}{2}$. Therefore, if $\dot{z} = \zeta(z)$:

$$(\dot{r}+ir\dot{\theta})e^{i\theta} = \frac{1}{1-2^{\frac{1}{2}-i\gamma-re^{i\theta}}}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^{\frac{1}{2}+i\gamma+re^{i\theta}}} = \zeta(\tfrac{1}{2}+i\gamma)+\zeta'(\tfrac{1}{2}+i\gamma)re^{i\theta}+O(r^2).$$

Since $\zeta(\frac{1}{2}+i\gamma)=0$ we can write

$$\frac{\dot{r}}{r}+i\dot{\theta} = \zeta'(\frac{1}{2}+i\gamma)+O(r).$$

Compute the derivative of $\zeta(z)$ at $r=0$, and use $\zeta(\frac{1}{2}+i\gamma)=0$ again to obtain

$$\lim_{r\to 0+}\Big(\frac{\dot{r}}{r}+i\dot{\theta}\Big) = \frac{-1}{1-2^{\frac{1}{2}-i\gamma}}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\langle n\rangle}{n^{\frac{1}{2}+i\gamma}}.$$

Finally, take the real and imaginary parts of both sides of this equation, and simplify, to obtain the two expressions given in the statement of this Proposition. ▮

Note that the correct form for the expressions for $\dot{r}/r$ and $\dot{\theta}$ in [2] should be:

$$\begin{aligned}
\frac{\dot{r}}{r} &= \frac{1}{rd}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^{\frac{1}{2}+r\cos\theta}}[\cos(\theta+\langle n\rangle(\gamma+r\sin\theta))\\
&\quad -2^{\frac{1}{2}-r\cos\theta}\cos(\theta+\langle\frac{n}{2}\rangle(\gamma+r\sin\theta)],\\
\dot{\theta} &= \frac{1}{rd}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^{\frac{1}{2}+r\cos\theta}}[-\sin(\theta+\langle n\rangle(\gamma+r\sin\theta))\\
&\quad +2^{\frac{1}{2}-r\cos\theta}\sin(\theta+\langle\frac{n}{2}\rangle(\gamma+r\sin\theta)], \text{ where}\\
d &:= 1+2^{1-2r\cos\theta}-2\cdot 2^{\frac{1}{2}-r\cos\theta}\cos(\langle 2\rangle(\gamma+r\sin\theta)).
\end{aligned}$$

PROPOSITION 0.2. *Let $\rho = \frac{1}{2} + i\gamma$ be a simple zero of $\zeta(z)$. Then ρ is a center for the flow $\dot{z} = \zeta(z)$ if and only if it is a Gram point. It is a node for the flow if and only if it is an anti Gram point. Otherwise ρ is a focus.*

Proof. Differentiating the functional equation in the form $\zeta(1-z) = \Lambda(z)\zeta(z)$ gives

$$-\overline{\zeta'(\rho)} = \Lambda(\rho)\zeta'(\rho)$$

Using [3, Lemma 2.1] and [1, Theorem 2.9] we can deduce that ρ is a Gram point if and only if $\Lambda(\rho) = 1$, if and only if $\zeta'(\rho)$ is pure imaginary, which is true if and only if ρ is a center.

The point ρ is an anti Gram point if and only if $\Lambda(\rho) = -1$, if and only if $\zeta'(\rho)$ is real, which is true if and only if ρ is a node. ▮

REFERENCES

1. K. A. Broughan, *Holomorphic flows on simply connected domains have no limit cycle*, Meccanica **38**(6) (2003), 699-709.
2. K. A. Broughan and A. Ross Barnett, *The holomorphic flow of the Riemann zeta function*, **73**, (2004), p987-1004.
3. K. A. Broughan and A. Ross Barnett, *Gram lines and the average of the real part of the Riemann zeta function*, (preprint).