Corrigendum for The holomorphic flow of the Riemann Zeta function

Kevin A. Broughan and A. Ross Barnett

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Theorem 4.5 of [2], describing the topological type of the zeros of the flow $\dot{s} = \zeta(s)$ at reflected points off the critical line, claiming they were the same, contains an error. We gratefully acknowledge Professor Cevat Gökcek for pointing out the error to us.

In some circumstances the types will never be the same. Indeed, it appears to be unlikely that a fixed formal relationship will exist between the types of flow for $\zeta(s)$ at reflected points off the critical line. Given that such points are not at hand (and expected not to be found), examples are currently impossible to come by. The revised statement and proof are given below. Some proof reading errors picked up by a helpful referee are also noted.

Let

$$\begin{aligned}
\Phi(s) &= \pi^{-s/2}\Gamma(\frac{s}{2}), \\
\Lambda(s) &= 2(2\pi)^{-s}\Gamma(s)\cos(\frac{\pi s}{2}), \\
\hat{\xi}(s) &= \Phi(s)\zeta(s), \\
\xi(s) &= \frac{s(s-1)}{2}\hat{\xi}(s).
\end{aligned}$$

Theorem 0.1. *Let $\zeta(s)$ have a zero of order m at $s = P_+ := \frac{1}{2} + x + iy$ for some $x, y > 0$. (a) Then the zero at the reflected point $P_- := \frac{1}{2} - x + iy$ for each function $\xi(s)$, $\hat{\xi}(s)$ and $\zeta(s)$ has the same local type if $m > 1$. (b) If $m = 1$ then, for $\xi(s)$, $\hat{\xi}(s)$ and $\zeta(s)$, both zeros are simple. (c) If $m = 1$, in addition, they have the same type for $\xi(s)$ and $\hat{\xi}(s)$. (d) Also, if the reflected points are centers or nodes for $\hat{\xi}(s)$ then $\arg\zeta'(P_+) \equiv \arg\zeta'(P_-) \bmod \pi$ is false.*

Proof. (a) Differentiate the functional equation in the form

$$\Phi(1-s)\zeta(1-s) = \Phi(s)\zeta(s)$$

m times to obtain

$$(-1)^m \Phi(\overline{P_-})\zeta^{(m)}(\overline{P_-}) = \Phi(P_+)\zeta^{(m)}(P_+).$$

If $m > 1$ the result follows from [1, Theorem 2.5], since $\Phi(s)$ is never zero.

(b) Now let $m = 1$. The equation now reads

$$-\Phi(\overline{P_-})\zeta'(\overline{P_-}) = \Phi(P_+)\zeta'(P_+).$$

It follows from this equation, since $\Phi(s)$ is entire and never zero, that the zeros at $P_\pm$ for $\xi(s)$, $\hat{\xi}(s)$ and $\zeta(s)$ are both simple.

(c) This follows by [1, Theorem 2.9], since $-\xi'(\overline{P_-}) = \xi'(P_+)$, with this same equation being true if $\xi(s)$ is replaced by $\hat{\xi}(s)$.

(d) Let P_+, P_- be centers or nodes for $\hat{\xi}(s)$. Then in the expression

$$\frac{\hat{\xi}'(P_+)}{\hat{\xi}'(P_-)} = \frac{\Phi(P_+)\zeta'(P_+)}{\Phi(P_-)\zeta'(P_-)}$$

the left hand side is real. But the difference of the imaginary parts of $\log\Gamma(z)$ at $P_+/2$ and $P_-/2$ is $\pi x/2 + O(1/y)$, so the ratio $\Phi(P_+)/\Phi(P_-)$ is never real, and so $\zeta'(P_+)/\zeta'(P_-)$ also cannot be real, and the result for (d) follows immediately. ∎

We would also wish to point out some proof reading errors. In the proof of Lemma 4.3, $\arg P_+$ and $\arg P_-$ should be $\arg\Gamma(P_+)$ and $\arg\Gamma(P_-)$ respectively. In the proof of Theorem 4.6, in the definition of C, s should be replaced by z, in the expression for the imaginary part of S, i should be omitted, and in the equation following (5), the exponent of each exponential should include i.

REFERENCES

1. K. A. Broughan, *Holomorphic flows on simply connected regions have no limit cycle*, Meccanica **38**(6) (2003), 699-709.
2. K. A. Broughan and A. R. Barnett, *The holomorphic flow of the Riemann zeta function*, Math. Comp. **73**(246) (2003), 987-1004.