# Flat primes and thin primes

Kevin A. Broughan and Zhou Qizhi

*University of Waikato, Hamilton, New Zealand*
*Version: 10th October 2008*
E-mail: kab@waikato.ac.nz, qz49@waikato.ac.nz

Flat primes and thin primes are primes where the shift by $\pm 1$ has a restricted form, namely a power of 2 or that times a square free number or odd prime respectively. They arise in the study of multi-perfect numbers. Here we show that the flat primes have asymptotic density relative to that of the full set of primes given by twice Artin's constant, that more than 50% of the primes are both lower and upper flat, and that the series of reciprocals of thin primes converges.

*Key Words:* flat prime, thin prime, twin prime, sieve.

MSC2000: 11A41, 11B05, 11B83.

## 1. INTRODUCTION

Some interesting subclasses of primes have been identified and actively considered. These include Mersenne primes, Sophie Germain primes, Fermat primes, Cullen's primes, Wieferich primes, primes of the form $n^2 + 1$, of the form $n! \pm 1$, etc. See for example [14, Chapter 5] and the references in that text. For any one of these classes, determining whether or not it is infinite has proved to be a very difficult problem.

In this article we explore two classes of primes, the so-called flat primes and the thin primes. They have simple representations, and we are able to get an idea of their densities relative to the full set of primes.

These primes are similar to primes of the form $k \cdot 2^e + 1$ considered by Erdös and Odlyzko, Chen and Sierpiński among others [5, 6, 16]. There the focus is mainly on the admissible values of odd integers $k$ with $k \leq x$, rather on the density of primes themselves having that structure. Erdös showed [5, Theorem 1] that the number $N(x)$ of odd numbers less than or equal to $x$ of the form $(p+1)/2^e$ satisfies

$$c_1 x \leq N(x) \leq c_2 x,$$

where $c_1$ and $c_2$ are positive absolute constants. In the opposite direction, a simple modification of the derivation of Sierpinski [16] gives an infinite number of integers $n$ (including an infinite set of primes) such that $n \cdot 2^e - 1$ is composite for every $e = 1, 2, 3, \cdots$.

DEFINITION 1.1. We say a natural number $n$ is a **flat number** if $n + 1 = 2^e$ or $n + 1 = 2^e q_1 \cdots q_m$ where $e \geq 1$ and the $q_i$ are distinct odd primes. If a prime $p$ is a flat number we say $p$ is a **(upper) flat prime**. Let

$$F(x) := \#\{p \leq x : p \text{ is a flat prime}\}.$$

Then it is straight forward to show that the density of flat numbers is the same as that of the odd square free numbers, i.e. the number of flat numbers up to $x$ is given by $4x/\pi^2 + O(\sqrt{x})$ [17].

DEFINITION 1.2. We say a natural number $n$ is a **thin number** if $n + 1 = 2^e q$ or $n + 1 = 2^e$ where $e \geq 1$ and $q$ is an odd prime. If a prime $p$ is a thin number we say $p$ is a **thin prime**. Let

$$T(x) := \#\{p \leq x : n \text{ is a thin prime}\}.$$

For example, among the first 100 primes, 75 primes are flat and among the first 1000 primes, 742 are flat. For thin primes the corresponding numbers are 38 and 213 respectively. The first 10 flat primes are 3, 5, 7, 11, 13, 19, 23, 29, 31, and 37. The first 10 thin primes are 3, 5, 7, 11, 13, 19, 23, 31, 37 and 43.

If $M(x)$ is the number of Mersenne primes up to $x$ then clearly, for all $x \geq 1$:

$$M(x) \leq T(x) \leq F(x) \leq \pi(x).$$

Figure 1 shows the ratio of $F(x)/\pi(x)$ over a small range. This gives some indication of the strength of Theorem 3.1 below - in the given range over 70% of all primes are flat.

Figure 2 shows the ratio of the number of thin primes up to $x$ to the number of twin primes up to $x$. The relationship between thin and twin primes comes from the method of proof of Theorem 4.1 below and there is no known direct relationship.

These types of number arise frequently in the context of multiperfect numbers, i.e. numbers which satisfy $k \cdot n = \sigma(n)$ where $\sigma(n)$ is the sum of the positive divisors of $n$. For example, when $k = 3$ all of the known

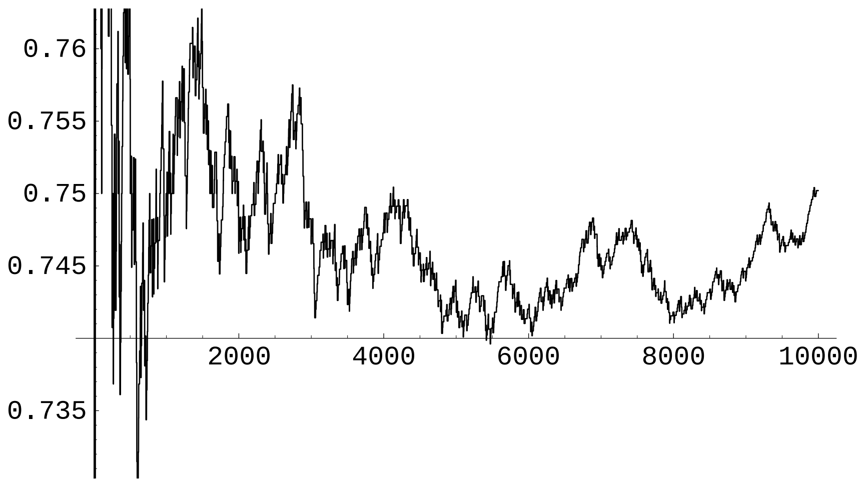


**FIG. 1.** The ratio $F(x)/\pi(x)$ for $1 \leq x \leq 10^4$.

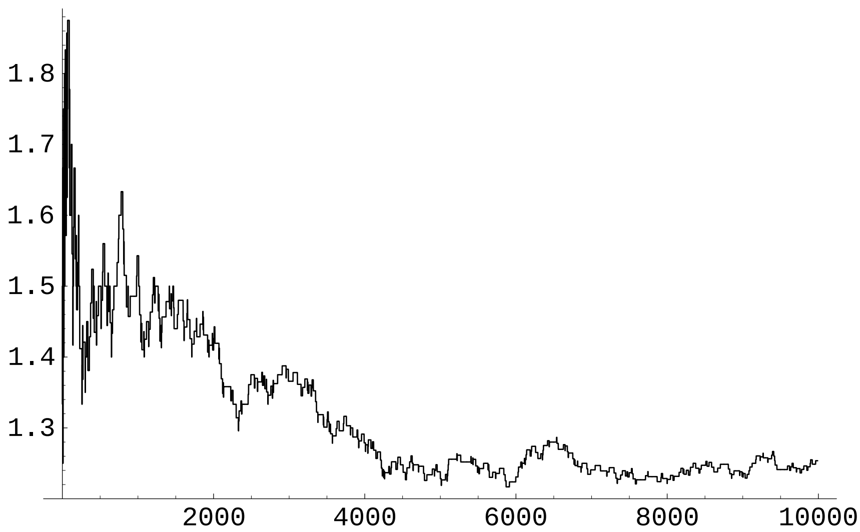


**FIG. 2.** The ratio of thin primes to twin primes up to $x$ for $1 \leq x \leq 10^4$.

examples of so-called 3-perfect numbers are

$$\begin{aligned}
c_1 &= 2^3 \cdot 3 \cdot 5, \\
c_2 &= 2^5 \cdot 3 \cdot 7, \\
c_3 &= 2^9 \cdot 3 \cdot 11 \cdot 31, \\
c_4 &= 2^8 \cdot 5 \cdot 7 \cdot 19 \cdot 37 \cdot 73, \\
c_5 &= 2^{13} \cdot 3 \cdot 11 \cdot 43 \cdot 127, \\
c_6 &= 2^{14} \cdot 5 \cdot 7 \cdot 19 \cdot 31 \cdot 151.
\end{aligned}$$

Each $c_i - 1$ is a flat number and each odd prime appearing on the right hand side is thin.

The paper is organized as follows: In Section 2 we first show that the asymptotic density of thin numbers up to $x$ is the same as that of the primes up to $x$. In Section 3 we show that the density of flat primes up to $x$, relative to the density of all primes, is given by $2A$ where $A$ is Artin's constant. A corollary to this is that there is a flat prime in every interval $[x,(1+\epsilon)x]$. This is followed by a demonstration that primes which are both lower and upper flat have density and constitute more than half of all primes. In Section 4 we then show that the thin primes are sufficiently sparse that the sum of their reciprocals converges. The final section is a numerical validation of what might be expected for the density of thin primes under the Bateman-Horn conjectures.

We use Landau's O,o,and $\ll$ notation. The symbols $p, q$ are restricted to be rational primes.

## 2. THIN AND FLAT NUMBERS

THEOREM 2.1. *The asymptotic density of thin numbers up to $x$ is the same as that of the primes up to $x$.*

*Proof.* The number of thin numbers up to $x$ is given by

$$N(x) = \sum_{n=1}^{\lfloor \frac{\log x}{\log 2} \rfloor} \pi\Big(\frac{x}{2^n}\Big) + O(\log x).$$

We will show that $\lim_{x\to\infty} N(x)/\pi(x) = 1$. To this end first consider a single term in the sum. By [15], there is a positive real absolute constant $\alpha$ such that for $x$ sufficiently large,

$$\frac{x}{\log x + \alpha} < \pi(x) < \frac{x}{\log x - \alpha}.$$

Therefore, for $n \in \mathbb{N}$ such that $1 \le n \le \frac{1}{4}\lfloor \frac{\log x}{\log 2} \rfloor$,

$$lb := \frac{1 - \frac{\alpha}{\log x}}{1 + \frac{\alpha}{\log x} - \frac{n \log 2}{\log x}} < \frac{2^n \pi(\frac{x}{2^n})}{\pi(x)} < \frac{1 + \frac{\alpha}{\log x}}{1 - \frac{\alpha}{\log x} - \frac{n \log 2}{\log x}} =: ub.$$

Clearly $lb$ and $ub$ tend to 1 as $x \to \infty$ for fixed $n$. The difference between the upper and lower bounds is

$$\begin{aligned} ub - lb &= \frac{1}{d}\Big(\frac{4\alpha}{\log x} - \frac{2\alpha n \log 2}{\log^2 x}\Big) \leq \frac{4\alpha}{d \log x}, \text{ where} \\ d &= \Big(1 - \frac{\alpha}{\log x} - \frac{n \log 2}{\log x}\Big)\Big(1 + \frac{\alpha}{\log x} - \frac{n \log 2}{\log x}\Big) \\ &= 1 - \frac{\alpha^2}{\log^2 x} + \frac{n^2 \log^2 2}{\log^2 x} - \frac{2n \log 2}{\log x} \geq \frac{1}{4} \text{ so} \\ ub - lb &\leq \frac{16\alpha}{\log x}. \end{aligned}$$

for $n$ in the given range and $x$ sufficiently large. Hence

$$\begin{aligned} &\Big| \sum_{n=1}^{\lfloor \frac{\log x}{\log 2} \rfloor /4} \frac{\pi(\frac{x}{2^n})}{\pi(x)} - 1 \Big| \\ &\leq \Big| \frac{\sum_{n \leq \frac{\log x}{\log \log x}} \pi(\frac{x}{2^n})}{\pi(x)} - \frac{\sum_{n \leq \frac{\log x}{\log \log x}} \frac{\pi(x)}{2^n}}{\pi(x)} \Big| + \sum_{n > \frac{\log x}{\log \log x}} \frac{1}{2^n} \\ &\leq \sum_{n \leq \frac{\log x}{\log \log x}} \frac{1}{2^n} \Big| \frac{\frac{2^n \pi(x)}{2^n}}{\pi(x)} - 1 \Big| + o(1) \\ &\leq \sum_{n \leq \frac{\log x}{\log \log x}} \frac{16\alpha}{2^n \log x} + o(1) \\ &\ll \frac{\log x}{\log x \log \log x} + o(1) = o(1) \end{aligned}$$

as $x \to \infty$.

For the remaining part of the summation range for $N(x)$, note that this corresponds to values of $x$ and $n$ which satisfy $1 \leq x/2^n \leq x^{3/4}$, so, using $\pi(x) \leq x/2$ and

$$S(x) := \sum_{n=\lfloor \frac{\log x}{\log 2} \rfloor /4}^{\lfloor \frac{\log x}{\log 2} \rfloor} \pi\Big(\frac{x}{2^n}\Big) \ll \frac{x^{\frac{3}{4}} \log x}{\log x} = O(x^{\frac{3}{4}})$$

it follows that $S(x)/\pi(x) \to 0$ as $x \to \infty$. Hence $N(x)/\pi(x) \to 1$. ∎

## 3. FLAT PRIMES

Theorem 3.1. *For all $H > 0$*

$$F(x) = 2\prod_p \Big(1 - \frac{1}{p(p-1)}\Big)\mathrm{Li}(x) + O\Big(\frac{x}{\log^H x}\Big),$$

*i.e. the relative density of flat primes is $2A = 0.7480\cdots$ where $A$ is Artin's constant.*

*Proof.* We begin following the method of Mirsky [10]. Fix $e \geq 1$ and let $x$ and $y$ satisfy $1 < y < x$ and be sufficiently large. Let $H > 0$ be the given positive constant. Define

$$F_e(x) := \#\{p \leq x : p \text{ is prime and } m \text{ square free such that } 2^e m = p + 1\}.$$

Then, if $\mu_2(m)$ is the characteristic function of the square free numbers,

$$\begin{aligned}
F_e(x) &= \sum_{\substack{p\leq x\\ p+1=2^e m}} \mu_2(m) = \sum_{p\leq x}\{\sum_{\substack{a:a\geq 1\\ a^2b2^e=p+1}} \mu(a)\}\\
&= \Sigma_1 + \Sigma_2, \text{ where}\\
\Sigma_1 &:= \sum_{p\leq x}\{\sum_{\substack{a:1\leq a\leq y\\ a^2b2^e=p+1}} \mu(a)\}, \text{ and}\\
\Sigma_2 &:= \sum_{p\leq x}\{\sum_{\substack{a>y\\ a^2b2^e=p+1}} \mu(a)\}.
\end{aligned}$$

Now using the Bombieri-Vinogradov theorem [4, Section 28] for the number of primes in an arithmetic progression, which is valid with a uniform error bound for the values of $e$ which will be needed:

$$\begin{aligned}
\Sigma_1 &= \sum_{a\leq y}\mu(a) \sum_{\substack{p:p\leq x\\ p\equiv -1 \bmod 2^e a^2}} 1\\
&= \sum_{a\leq y}\mu(a)\Big(\frac{\mathrm{Li}(x)}{\phi(2^e a^2)} + O(\frac{x}{\log^{2H+1} x})\Big)\\
&= \Big(\sum_{a\geq 1}\frac{\mu(a)}{\phi(2^e a^2)})\Big)\mathrm{Li}(x) + O\Big(\frac{x}{\log x}\sum_{a>y}\frac{1}{\phi(2^e a^2)}\Big) + O\Big(\frac{xy}{\log^{2H+1} x}\Big).
\end{aligned}$$

Note that the function $g(n) := 2^{1-e}\phi(2^e n^2)$ is multiplicative, so the coefficient of $\mathrm{Li}(x)$ may be rewritten

$$\begin{aligned}\frac{1}{2^{e-1}}\sum_{a\geq 1}\frac{2^{e-1}\mu(a)}{\phi(2^e a^2)} &= \frac{1}{2^{e-1}}\prod_p\Big(1-\frac{2^{e-1}}{\phi(2^e p^2)}\Big)\\ &= \frac{1}{2^{e-1}}\frac{3}{4}\prod_{p \text{ odd}}\Big(1-\frac{1}{p^2-p}\Big)=\frac{3A}{2^e}.\end{aligned}$$

Now consider the sum in the first error term:

$$\sum_{a>y}\frac{1}{\phi(2^e a^2)} \leq \sum_{a>y}\frac{1}{2^e\phi(a^2)} \ll \frac{1}{2^e}\sum_{a>y}\frac{\log\log a}{a^2}.$$

Therefore

$$O\Big(\frac{x}{\log x}\sum_{a>y}\frac{1}{\phi(2^e a^2)}\Big)=O\Big(\frac{x\log\log y}{2^e y\log x}\Big).$$

For the second sum:

$$|\Sigma_2| \leq \sum_{p<x}\{\sum_{\substack{a>y\\ p+1=2^e a^2 b}} 1\} \leq \sum_{\substack{a>y\\ 2^e a^2 b\leq x}} 1 = O\Big(\frac{x}{2^e y}\Big),$$

and therefore

$$F_e(x)=\frac{3A}{2^e}\mathrm{Li}(x)+O\Big(\frac{x\log\log y}{2^e y\log x}\Big)+O\Big(\frac{x}{2^e y}\Big)+O\Big(\frac{xy}{\log^{2H+1}x}\Big).$$

If we choose $y=\log^H x$, then

$$F_e(x)=\frac{3A}{2^e}\mathrm{Li}(x)+O(\frac{x}{\log^{H+1}x}).$$

Now let

$D_e(x) := \#\{p\leq x : p \text{ is prime}, p+1=2^e m, \text{ with } m \text{ square free and odd}\}$.

By, [10, Theorem 2], $D_1(x)=A\cdot\mathrm{Li}(x)+O\big(\frac{x}{\log^{H+1}x}\big)$. Considering the even and odd cases, for all $e\geq 1$, we have $F_e(x)=D_e(x)+D_{e+1}(x)$ so

$$F_1(x)+F_2(x)+\cdots = D_1(x)+2(D_2(x)+D_3(x)+\cdots)$$

and therefore

Then

$$
\begin{aligned}
F(x) &= \sum_{e=1}^{\frac{\log x}{\log 2}} D_e(x) + O(\log x) \\
&= \frac{1}{2}(D_1(x) + F_1(x) + F_2(x) + \cdots) + O(\log x) \\
&= \frac{A}{2}(1 + \frac{3}{2^1} + \frac{3}{2^2} + \cdots)\mathrm{Li}(x) + O\big(\frac{x}{\log^{H+1} x}\big) \\
&= 2A\mathrm{Li}(x) + O\Big(\frac{x}{\log^H x}\Big)
\end{aligned}
$$

and this completes the proof. ▌

Note that if we call primes with the shape $p = 2^e p_1 \cdots p_m + 1$ **lower flat**, their density is the same as that of the flat primes, so more than 20% of all primes are both flat and lower flat. However, this figure very significantly underestimates the proportion of such primes - see Theorem 3.2 and its corollary below. By analogy, flat primes are also called **upper flat primes**.

Corollary 3.1. *For all $\epsilon > 0$ and $x \geq x_\epsilon$ there exists a flat prime in the interval $[x, (1+\epsilon)x]$.*

Note also that it would be possible to adapt the method of Adleman, Pomerance and Rumley [1, Proposition 9] to count flat primes in arithmetic progressions.

Theorem 3.2. *Let the constant $H > 0$ and the real variable $x$ be sufficiently large. Let the set of primes which are both lower and upper flat which are less than $x$ be given by*

$$
\begin{aligned}
B(x) &= \{p \leq x : \exists e \geq 1,\ f \geq 1 \text{ and odd square free } u, v \text{ so } p - 1 = 2^e v, \\
&\quad p + 1 = 2^f u\}.
\end{aligned}
$$

*Then*

$$B(x) = A_2 \mathrm{Li}(x) + O(\frac{x}{\log^H x})$$

*where the constant*

$$A_2 = \prod_{p \text{ odd}} \big(1 - \frac{2}{p^2 - p}\big) = 0.53538 \cdots .$$

*Proof.* Let $e, f \geq 1$ and define the sets:

$$\begin{aligned} L_e &:= \{p \leq x : \exists \text{ odd square free } v \text{ so } p-1 = 2^e v\} \\ U_f &:= \{p \leq x : \exists \text{ odd square free } u \text{ so } p+1 = 2^f u\}. \end{aligned}$$

Then $L_1 \cap U_1 = \emptyset$ and $L_e \cap U_f = \emptyset$ for all $e \geq 2, f \geq 2$ so we can write

$$B(x) = \{\cup_{f\geq 2} L_1 \cap U_f\} \cup \{\cup_{e \geq 2} U_1 \cap L_e\}$$

where all of the unions are disjoint.

Now fix $e \geq 2$. We will first estimate the size of $U_1 \cap L_e$,where

$$U_1 \cap L_e = \{p \leq x : \exists \text{ odd square free } u, v \text{ so } p+1 = 2u, p-1 = 2^e v\}.$$

Then

$$\begin{aligned} \#U_1 \cap L_e &= \sum_{p \leq x} \{ \sum_{\substack{p+1=2u, \\ p-1=2^e v, \\ u,v \text{ odd and square free}}} 1\} \\ &= \sum_{p\leq x} \{ \sum_{\substack{a,b \ odd, (a,b)=1, \\ p \equiv -1 \bmod a^2, \\ p\equiv 1 \bmod b^2, \\ p \equiv 1+2^e \bmod 2^{e+1}}} \mu(a)\mu(b)\} \\ &= \sum_{p\leq x} \{ \sum_{\substack{a,b \ odd, (a,b)=1, \\ p\equiv u \bmod 2^{e+1} a^2 b^2}} \mu(a)\mu(b)\} \\ &= \sum_{p \leq x} \tau^*(d) \{ \sum_{\substack{d \ odd, \\ p \equiv u \bmod 2^{e+1} d^2}} \mu(d)\} \end{aligned}$$

where $u$, the residue obtained through an application of the Chinese Remainder Algorithm, is dependent on $d$ and $e$, and $\tau^*(d)$ is the number of unitary divisors of $d$, a multiplicative function with $\tau^*(p) = 2$.

We then split and reverse the sum in a similar manner as in the proof of Theorem 3.1 to arrive at

$$\begin{aligned} \#U_1 \cap L_e &= \Big( \sum_{d \geq 1, d \ odd} \frac{\tau^*(d)}{\phi(2^{e+1}d^2)} \Big) \mathrm{Li}(x) + O\big(\frac{x}{\log^{H+1} x}\big) \\ &= \frac{1}{2^e} \prod_{p \ odd} \big(1 - \frac{2}{p^2-p}\big) \mathrm{Li}(x) + O\big(\frac{x}{\log^{H+1} x}\big) \end{aligned}$$

Summing over $e \geq 2$ and, noticing that the sizes for each corresponding $L_1 \cap U_e$ are the same, we obtain the stated value of $B(x)$. ▮

Figure 3 compares the number of primes up to $80,000$ with the number of primes up to $80,000$ which are both lower and upper flat.

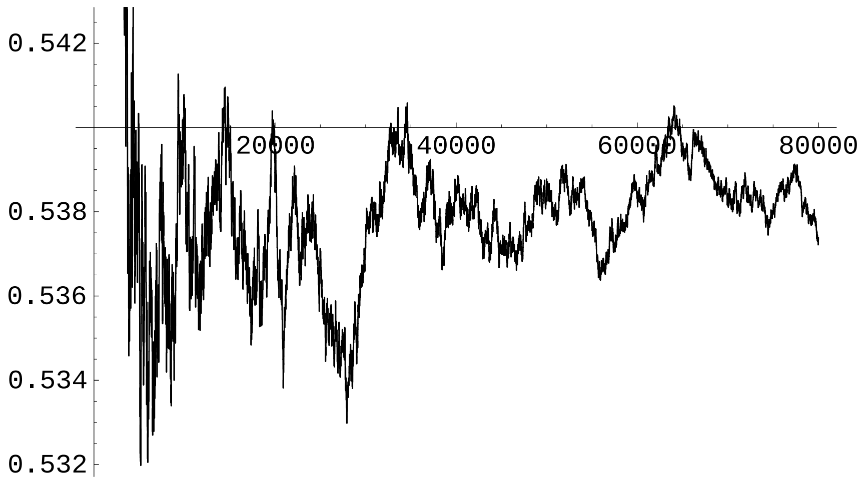


**FIG. 3.** The ratio $B(x)/\pi(x)$ for $1 \leq x \leq 8 \cdot 10^4$.

COROLLARY 3.2. *It follows from Theorems 3.1 and 3.2 that the set of rational primes may be divided into 4 disjoint classes: those both lower and upper flat - about 54%, those either lower or upper flat but not both - each about 21%, and those neither upper nor lower flat - 4%.*

## 4. THIN PRIMES

In the paper [19, Theorem 3] a proof is set out for a result given below on the number of primes up to $x$ giving a lower bound for the number primes with fixed consecutive values of the number of distinct prime divisors of shifts of the primes by $a$, with the parameter $a$ having the explicit value 2. It is remarked that a similar proof will work for all integer (non-zero) $a$. Here is the statement taken from Mathematical Reviews (although the lower bound for $m$ is not given):

Let $a$ be a non-zero integer and (for $m \geq 1$) define

$$\mathcal{P}(m, x, \omega) := \#\{p : p \leq x, \omega(p + a) = m\}.$$

Then there exist positive absolute constants $b$ and $c$ such that as $x \to \infty$

$$\mathcal{P}(m,x,\omega) + \mathcal{P}(m+1,x,\omega) \geq c\frac{x(\log\log x)^{m-1}}{(m-1)!\log^2 x}$$

holds for $1 \leq m \leq b\log\log x$.

If we use the result in case $a = 1$, we are able to show the number of thin primes is infinite.

To see this let $a = 1$, $m = 1$ and $x$ be sufficiently large. Then

$$\begin{aligned} T(x) + M(x) &= \mathcal{P}(1,x,\omega) + \mathcal{P}(2,x,\omega) \\ &= \#\{p \leq x : p+1 = 2^e \text{ or } p+1 = 2^e q^f, e \geq 1, f \geq 1, \text{ or } p = 2\} \end{aligned}$$

where

$$M(x) := \#\{p \leq x : p+1 = 2^e q^f, e \geq 1, f \geq 2\}.$$

Then

$$\begin{aligned} M(x) &\leq \sum_{e=1}^{\log x}\sum_{f=2}^{\log x} \pi((\frac{x}{2^e})^{\frac{1}{f}}) + O(\log x) \\ &\ll \log x \sum_{e=1}^{\log x} \pi(\sqrt{\frac{x}{2^e}}) \\ &\ll \log^2 x\ \pi(\sqrt{x}) \ll \sqrt{x}\log x \end{aligned}$$

Therefore, by the quoted result above, the number of thin primes less than or equal to $x$ is bounded below by a constant times $x/\log^2 x$, so must be infinite.

However there are parts of the proof of [19, Theorem 3] that do not appear to work, even for the given case $a = 2$, and, in addition, the implied lower bound should be $m \geq 2$. Apparently the best available safe result, using the method of Chen, appears to be that of Heath-Brown [8, Lemma 1] from which we can easily show that the number of primes $H(x)$ such that $p \leq x$ and either $p+1 = 2p_1$ or $p+1 = 2p_1p_2$, with the $p_i$ being odd primes, is infinite, indeed $H(x) \gg x/\log^2 x$.

Based on this evidence, the Bateman-Horn conjecture set out in Section 5 below, and numerical evidence, we are led to the conjecture:

**Conjecture:** The number of thin primes up to $x$ satisfies

$$T(x) \gg \frac{x}{\log^2 x}.$$

The order of difficulty of this conjecture appears to be similar to that there are an infinite number of twin primes or Sophie Germain primes. As usual upper bounds are much easier to obtain.

THEOREM 4.1. *As $x \to \infty$*

$$T(x) \ll \frac{x}{\log^2 x}.$$

*Proof.* First let $e \geq 1$ be fixed and apply the sieve of Brun in the same manner as for the classical twin primes problem (for example [18, Theorem 4]) or [3, Theorem 13.1]) to count

$$J_e(x) := \#\{p \leq x : 2^e p - 1 \text{ is prime}\},$$

noting that if

$$\mathcal{A} = \{m(2^e m - 1) : m \leq x\}$$

and $\rho(d)$ is the number of solutions modulo $d$ which satisfy

$$m(2^e m - 1) \equiv 0 \bmod d,$$

then $\rho$ is a multiplicative function, $\rho(2) = 1$ and $\rho(p) = 2$ for odd primes $p$ leading to the same bound as in the twin primes problem, namely

$$J_e(x) \ll \frac{x}{\log^2 x}.$$

Now we use the fact, proved using induction for $m \geq 4$, that, for all $m \geq 1$,

$$\sum_{n=1}^{m} \frac{2^n}{n^2} < 5\frac{2^m}{m^2} \qquad (2).$$

(For sufficiently large $m$, the 5 can be replaced by $2 + \epsilon$ but we don't need this.)

Finally, let $x$ be large and choose $m \in \mathbb{N}$ so $2^m \le x < 2^{m+1}$. Then

$$\begin{aligned}
T(x) &= \sum_{e=1}^{\lfloor \frac{\log x}{\log 2} \rfloor} \big(J_e(\frac{x}{2^e}) + O(1)\big) \\
&\ll \sum_{e=1}^{\lfloor \frac{\log x}{\log 2} \rfloor - 1} \frac{x}{2^e} \frac{1}{\log^2 \frac{x}{2^e}} + O(\log x) \\
&\le \sum_{e=0}^{\lfloor \frac{\log x}{\log 2} \rfloor - 1} \frac{x}{2^e} \frac{1}{\log^2 \frac{x}{2^e}} + O(\log x) \\
&\le \sum_{e=0}^{\lfloor \frac{\log 2^{m+1}}{\log 2} \rfloor - 1} \frac{2^{m+1}}{2^e} \frac{1}{\log^2 \frac{2^{m+1}}{2^e}} + O(\log x) \\
&= \frac{1}{\log^2 2} \sum_{e=0}^{m} \frac{2^{m-e+1}}{(m+1-e)^2} + O(\log x) \\
&= \frac{1}{\log^2 2} \sum_{n=1}^{m+1} \frac{2^n}{n^2} + O(\log x) \\
&< 5 \frac{1}{\log^2 2} \frac{2^{m+1}}{(m+1)^2} + O(\log x) \text{ by (2)} \\
&< 10 \frac{1}{\log^2 2} \frac{2x \log^2 2}{\log^2 x} + O(\log x) \\
&\ll \frac{x}{\log^2 x},
\end{aligned}$$

completing the proof of the theorem. ▮

So the asymptotic bound is the same as that for twin primes. In the same manner as originally derived by Brun for the sum of reciprocals of the twin primes (for example [12, Theorem 6.12]) we obtain:

Corollary 4.1. *The sum of the reciprocals of the thin primes is finite.*

*Proof.* If $p_n$ is the $n$'th thin prime then, by Theorem 4.1,

$$\begin{aligned}
n &= T(p_n) \ll \frac{p_n}{\log^2 p_n} \\
&\ll \frac{p_n}{(\log n)^2} \text{ so} \\
\frac{1}{p_n} &\ll \frac{1}{n \log^2 n}.
\end{aligned}$$

$\blacksquare$

## 5. HARDY-LITTLEWOOD-BATEMAN-HORN CONJECTURES

The well known Hardy-Littlewood-Bateman-Horn conjectures [7, 3] give an asymptotic formula for the number of simultaneous prime values of sets of polynomials in $\mathbb{Z}[x]$, with some restrictions on the polynomials. In the case of twin primes the polynomials are $f_o(x) = x,\ f_1(x) = x + 2$ and if

$$\pi_2(x) := \#\{p \le x : p + 2 \text{ is prime}\}$$

then the formula predicted is

$$\pi_2(x) \sim 2C_2 \int_2^x \frac{du}{\log^2 u}$$

where $C_2$ is the so-called twin prime constant [13] defined by

$$C_2 := \prod_{p>2} \Big(1 - \frac{1}{(p-1)^2}\Big).$$

In the case of thin primes the conjectures only apply to forms with fixed $e \ge 1$ with polynomials $f_0(x) = x,\ f_e(x) = 2^e x - 1$. If

$$T_e(x) := \#\{p \le x : p + 1 = 2^e q\}$$

Then the formulas predict

$$T_e(x) \sim \frac{2C_2}{2^e} \int_2^x \frac{du}{\log^2 u}.$$

The factor $1/2^e$ occurs simply because $p \le x + 1$ if and only if $q \le x/2^e$. Hence

$$\frac{T(x)}{\pi_2(x)} \sim \frac{\sum_{e=1}^{\log x/\log x} T_e(x)}{\pi_2(x)} \sim 1.$$

To test this numerically we evaluated the ratio of the number of thin primes up to $x$ to the number of twin primes up to $x$ for $x$ up to $4 \times 10^6$ in steps of $10^5$ and obtained the following values:

```
{1., 1.20343, 1.16852, 1.17134, 1.16036, 1.15882, 1.14819, 1.1447,
1.14499,  1.1428, 1.13515, 1.12896, 1.12543, 1.1234, 1.11715,
```

```
1.1184, 1.11729, 1.11438,  1.11168, 1.1099, 1.11169, 1.1106,
1.11125, 1.11095, 1.11221, 1.11317, 1.1134,  1.11251, 1.1118,
1.11306, 1.11179, 1.11015, 1.10986, 1.1096, 1.10876,  1.10924,
1.10912, 1.10676, 1.10623, 1.10536}.
```

demonstrating some convergence towards the predicted value 1. If the relationship between the thin and twin primes could be made explicit this would assist in a proof of the twin primes conjecture.

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