A conjecture of De Koninck regarding particular square values of the sum of divisors function

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Abstract

We study integers $n > 1$ satisfying the relation $\sigma(n) = \gamma(n)^2$, where $\sigma(n)$ and $\gamma(n)$ are the sum of divisors and the product of distinct primes dividing n, respectively. If the prime dividing a solution n is congruent to 3 modulo 8 then it must be greater than 41, and every solution is divisible by at least the fourth power of an *odd* prime. Moreover at least $2/5$ of the exponents a of the primes dividing any solution have the property that $a + 1$ is a prime power. Lastly we prove that the number of solutions up to $x > 1$ is at most $x^{1/6+\epsilon}$, for any $\epsilon > 0$ and all $x > x_\epsilon$.

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1 Introduction

A decade ago, Jean-Marie De Koninck asked for all integer solutions n to the equation

$$\sigma(n) = \gamma(n)^2 \tag{1}$$

where $\sigma(n)$ denotes the sum of all positive divisors of n, and $\gamma(n)$ is the product of the distinct prime divisors of n. The only known solutions with $1 \leq n \leq 10^{11}$ are $n = 1$ and $n = 1782$, and so De Koninck sensibly conjectured that there exist no other solutions. It is included in Richard Guy's compendium [4, Section B11] as an unsolved problem.

In [2] a number of restrictions on the form of Equation (1) were developed: the two solutions $n = 1$ and $n = 1782$ are the only ones having $\omega(n) \leq 4$; furthermore, if an integer $n > 1$ is fourth power free (i.e. $p^4 \nmid n$ for all primes p), then it was shown that n cannot satisfy De Koninck's equation.

The aim of this work is to present further items of evidence in support of De Koninck's conjecture, and to indicate the necessary structure of a hypothetical counter-example. In fact, upon combining together the results of [2] and this article, then any non-trivial solution other than 1782 must be even, have one prime divisor to power 1 and possibly another prime divisor to a power congruent to 1 modulo 4, while all other odd prime divisors should occur only to even powers. Here we shall establish that if the prime to power 1 is congruent to 3 modulo 8, then it must be no less than 43 (Proposition 1). Moreover, we prove that at least one odd prime divisor must appear with an exponent no smaller than 4 (Theorem 1).

Applying an idea from [3], we show in Corollary 2 that more than 2/5 of the exponents a appearing in the prime factorization of any solution of Equation (1) are such that $a+1$ is a prime or a prime power. We then count the number of potential solutions n up to x, in the following manner: using results of Pollack and Pomerance [8], and by extending a method of [2, Thm 1], we shall prove in Theorem 2 that the number of solutions $n \leq x$ to Equation (1) can be at most $x^{1/6+\epsilon}$, for any $\epsilon > 0$ and every $x > x_\epsilon$.

Finally, by exploiting the properties of the product compactification of $\mathbb{N}$, we show there are only finitely many solutions to (1) supported on any given finite set of primes $\mathcal{P}$. Indeed we will prove a more general result for the equation

$$\sigma(n)^\alpha \times \phi(n)^\beta \;=\; \theta \times n^\mu \times \gamma(n)^\tau \tag{2}$$

where $\alpha, \beta, \mu, \tau \in \mathbb{Z}$ with $\theta > 0$ some fixed rational, and $\alpha + \beta > \mu$ (see Theorem 3). The argument itself has a rather different flavour from that in [5].

Notations: If p is prime then $v_p(n)$ is the highest power of p which divides n, $\omega(n)$ will denote the number of distinct prime divisors of n, and $\mathcal{K}$ is the set of all solutions to $\sigma(n) = \gamma(n)^2$. Lastly, the symbols p, q, p_i, q_i are reserved exclusively for odd primes.

2 Preliminary Lemmas

We begin by recalling some basic structure theory concerning solutions to Equation (1). The following two background results were proved in [2].

Lemma 1. *If $n > 1$ belongs to $\mathcal{K}$, then one has a decomposition*

$$n \ = \ 2^e \times p_1 \times \prod_{i=2}^{s} p_i^{a_i}$$

where $e \geq 1$, and a_i is even for all $i = 3, \ldots, s$. Furthermore, either a_2 is even in which case $p_1 \equiv 3 \pmod 8$, or instead $a_2 \equiv 1 \pmod 4$ and $p_1 \equiv p_2 \equiv 1 \pmod 4$.

Lemma 2. *If $n > 1$ is an element of $\mathcal{K}$ and does not equal to $1782 = 2 \cdot 3^4 \cdot 11$, then n has at least 5 distinct prime factors, and there exists a prime (either even or odd) dividing n to at least a fourth power.*

The proof of the next result is due Pollack, and can be found in [6].

Lemma 3. *If $\sigma(n)/n = N/D$ with $\gcd(N, D) = 1$, then given $x \geq 1$ and $d \geq 1$:*

$$\#\big\{n \leq x \text{ such that } D = d\big\} = \ x^{o(1)} \quad \text{as } x \to \infty.$$

Lastly we will require Apéry's solution to the generalized Ramanujan-Nagel equations.

Lemma 4. *(Apéry [1]) The Diophantine equation $x^2 + D = 2^{n+2}$, with given non-zero integer $D \neq 7$, has at most two solutions. In addition:*

(i) if $D = 23$ then $(x, n) \in \big\{(3, 5), (45, 11)\big\}$,

(ii) if D has the form $2^m - 1$ with $m \geq 4$, then $(x, n) \in \big\{(1, m), (2^m - 1, 2m - 1)\big\}$.

Hence, in both these cases, there are exactly two solutions.

3 Restrictions on primes dividing members of $\mathcal{K}$

In this section, we shall make a preliminary study of restrictions on the possible values of p_1 and p_2 associated to elements of $\mathcal{K}$, additional to those described in Lemma 1 above. Clearly $p_1 + 1$ cannot be divisible by any cube, otherwise Equation (1) is violated. Hence for prime numbers congruent to 3 modulo 8, this excludes first 107 and secondly (in increasing order) 499 from occurring.

We will henceforth refer to these as **bad De Koninck primes**; indeed there are an infinite number of primes $p \equiv 3 \pmod 8$ such that $p+1$ is divisible by a proper cube. In Proposition 1 below, we shall prove that 3, 11 and 19 are also bad. In the case $p_1 \equiv p_2 \equiv 1 \pmod 4$ and $a_2 = 1$, this same constraint applied to $(p_1+1)(p_2+1)$ excludes for those primes less than 100, the pairs

$$\{5,17\},\ \{5,53\},\ \{5,89\},\ \{13,53\},\ \{13,97\},\ \{17,29\},\{17,41\},\ \{17,53\},$$
$$\{17,89\},\ \{29,53\},\ \{29,89\},\ \{37,53\},\ \{41,53\},\ \{41,89\},\ \{41,97\}$$

called here **bad De Koninck pairs**. Later in Corollary 1, we show $\{5,13\}$ is also bad.

Proposition 1. *Under the same notations as Lemma 1, if a solution $n \in \mathcal{K}$ satisfies both $\omega(n) > 4$ and $p_1 \equiv 3 \pmod 8$, then the prime $p_1 \geq 43$.*

Proof. First suppose that $p_1 = 3$, in which case

$$(2^{e+1}-1) \times 4 \times \prod_{i=2}^{m} \frac{\sigma(p_i^{a_i})}{p_i^2} \;=\; 4 \times 3^2.$$

As a direct consequence $2^{e+1}-1 < 9$ so $e \in \{1,2\}$, and by [2, Theorem 3] we can assume $a_2 \geq 4$. Therefore

$$3 \times 13 \;=\; 3(3^2+3+1) \;<\; (2^{e+1}-1) \times \frac{\sigma(p_2^{a_2})}{p_2^2} \;<\; 3^2,$$

which is obviously false, and we conclude that $p_1 \neq 3$.

Next if one supposes that $p_1 = 11$, then

$$(2^{e+1}-1) \times 12 \times \prod_{i=2}^{m} \sigma(p_i^{a_i}) \;=\; 4 \times 11^2 \times \prod_{i=2}^{m} p_i^2,$$

thus $3 \times (2^{e+1}-1) < 11^2$ which implies that $1 \leq e \leq 4$. If all of the a_i were strictly less than 4, then by [2, Theorem 3] again we would have $e = 4$, in which case

$$(2^{e+1}-1) \times (p_1+1) \times \prod_{i=2}^{m} \frac{\sigma(p_i^{a_i})}{p_i^2} \;=\; 4 \times 11^2.$$

The latter implies

$$31 \times 3 \times \prod_{i=2}^{m} \frac{\sigma(p_i^{a_i})}{p_i^2} \;=\; 11^2,$$

hence there exists an $i \geq 2$ with $11 \mid p_i^2 + p_i + 1$; this is impossible since $11 \not\equiv 1 \bmod 3$. It follows there is at least one $i \geq 2$ with $a_i \geq 4$, and without loss of generality suppose that it is a_2 say. One therefore obtains an inequality

$$(2^{e+1} - 1) \times \frac{p_1 + 1}{4} \times \frac{\sigma(p_2^{a_2})}{p_2^2} \; < \; 11^2$$

and consequently,

$$3^2 \times \frac{\sigma(3^4)}{3^2} \; = \; 11^2 \; < \; 11^2$$

which is clearly false. Therefore $p_1 \neq 11$.

Finally suppose $p_1 = 19$. Using Lemma 1 we can write $n = 2^e \times p_1 \times \prod_{i=2}^m p_i^{a_i}$, whence

$$(2^{e+1} - 1) \times (p_1 + 1) \times \prod_{i=2}^{m} \frac{\sigma(p_i^{a_i})}{p_i^2} \; = \; 4p_1^2.$$

Thus $(2^{e+1} - 1) \times 5F = 19^2$ where F is a positive rational value strictly greater than 1. As a consequence $(2^{e+1} - 1) < 19^2/5$, implying that $1 \leq e \leq 5$.

Case (1): If $e = 5$ then

$$9 \times 7 \times 5F \; = \; 19^2$$

and it follows that $F < 19^2/315 < 1.15$. If some exponent $a_i \geq 3$ then $F \geq \sigma(p_i^3)/p_i^2 > 3$ which cannot occur, and therefore one may assume that $a_i = 2$ for every $i \in \{2, \ldots m\}$. Now by studying the left hand side, there must exist a prime p_i (which we will call p_2) that equals 3. Then $\sigma(p_2^2) = 3^2 + 3 + 1 = 13$ yields a new prime, denoted p_3, with $\sigma(p_3^2) = 13^2 + 13 + 1 = 3 \times 61$. One thereby obtains a left hand side with at least three 3's in the numerator but at most two 3's in the denominator, while the right hand side has none. This contradiction shows $e < 5$.

Case (2): If $e = 4$ then

$$31 \times 5 \times \prod_{i=2}^{m} \frac{\sigma(p_i^{a_i})}{p_i^2} \; = \; 19^2.$$

Arguing as in the previous case, without loss of generality assume $a_i = 2$ for $i \geq 2$. Examining the left hand side, one of the primes p_i must equal 31; let us call it p_2. Then we have $\sigma(p_2^2) = 31^2 + 31 + 1 = 3 \times 331$, thence the new prime $p_3 = 331$ gives $\sigma(p_3^2) = 331^2 + 331 + 1 = 3 \times 7 \times 5233$, and ultimately $p_4 = 7$ with $7^2 + 7 + 1 = 3 \times 19$. Hence there are at least three 3's in the numerator and exactly two in the denominator, with none occurring on the right hand side. This shows $e < 4$.

Case (3): If $e = 3$ then $15 \times 5F = 19^2$ implies $75 \times (3^2 + 3 + 1) < 19^2$, which is false.

Case (4): If $e = 2$ then we would get $7 \times 5 \times 13 < 19^2$, which again is false.

Case (5): Henceforth we consider the situation where $e = 1$. It follows that

$$3 \times 5 \times \prod_{i=2}^{m} \sigma(p_i^{a_i}) \;=\; 19^2 \times \prod_{i=2}^{m} p_i^2$$

implying $3 \mid n$. One can then take $p_2 = 3$, and (by Lemma 1) assume that a_2 is even. Suppose first that $a_2 \geq 6$. Then $\sigma(3^{a_2}) \geq \sigma(3^6) = 1093$, in which case

$$5 \times 1093 \times \prod_{i=3}^{m} \sigma(p_i^{a_i}) \;\leq\; 5 \times \sigma(3^{a_2}) \times \prod_{i=3}^{m} \sigma(p_i^{a_i}) \;=\; 19^2 \times 3 \times \prod_{i=3}^{m} p_i^2$$

which is false, whence $a_2 \in \{2, 4\}$. However if $a_2 = 2$, then

$$3 \times 5 \times (3^2 + 3 + 1) \times \prod_{i=3}^{m} \frac{\sigma(p_i^{a_i})}{p_i^2} \;=\; 19^2 \times 3^2$$

and there must exist an odd prime dividing n which is greater than 3, and which divides n to a power not less than 4. This eventuality in turn implies

$$5 \times 13 \times (5^2 + 5 + 1) \;<\; 19^2 \times 3$$

which again is impossible.

Hence the only remaining possibility is that $a_2 = 4$. Because $\sigma(19) = 2^2 \times 5$ and $\sigma(3^4) = 11^2$, one may then assume $p_3 = 5$ and $p_4 = 11$, which gives us the equality

$$\sigma(2^1)\sigma(19^1)\sigma(3^4)\sigma(5^{a_3})\sigma(11^{a_4}) \times \prod_{i=5}^{m} \frac{\sigma(p_i^{a_i})}{p_i^2} \;=\; 2^2 \times 19^2 \times 3^2 \times 5^2 \times 11^2.$$

Canceling like terms yields

$$5^{a_3} \times 11^{a_4} \;<\; \sigma(5^{a_3})\sigma(11^{a_4}) \;\leq\; 19^2 \times 3 \times 5$$

which is false if either $a_3 \geq 4$ or $a_4 \geq 4$; since both are even, clearly $a_3 = a_4 = 2$.

Now $2 \cdot 19 \cdot 3^4 \cdot 5^2 \cdot 11^2 \notin \mathcal{K}$ so there exists a prime $p_5 \geq 7$ such that $p_5^{a_5} \| n$ with a_5 even. If $a_5 \geq 4$ then one would have

$$(7^2 + 7) \times 5^2 \times 11^2 \;<\; 5^2 \times 11^2 \times \frac{\sigma(p_5^{a_5})}{p_5^2} \;<\; 19^2 \times 3 \times 5$$

which is certainly false; thus all primes other than $2, 3, 19$ which divide n must do so exactly to the power 2.

As a consequence $m \geq 5$, and we can write

$$n \;=\; 2 \times 19 \times 3^4 \times 5^2 \times 11^2 \times \prod_{i=5}^{m} p_i^2 .$$

Substituting this form into the equation $\sigma(n) = \gamma(n)^2$ and then canceling, one deduces

$$31 \times 131 \times \prod_{i=5}^{m} \left(\frac{p_i^2 + p_i + 1}{p_i^2} \right) \;=\; 19^2 \times 3 \times 5 .$$

Therefore the set of p_i with $5 \leq i \leq m$ includes $\{31, 131\}$ and none out of $\{3, 5, 19\}$. However $\sigma(31^2) = 3 \times 331$, $\sigma(131^2) = 17293$ and $\sigma(17293^2) = 3 \cdot 13 \cdot 7668337$, hence $3^2 = 9$ divides the numerator of the product on the left and does not cancel with any denominator. This circumstance is impossible, as 9 does not divide the right hand side.

The above contradiction completes the proof that $p_1 \neq 19$. □

Proposition 2. *If $p_1 \equiv 1 \pmod 4$ and $a_2 \geq 5$, then $p_1 \geq 173$.*

Proof. Applying Lemma 1 one knows $p_2 \geq 5$, and we can write

$$(2^{e+1} - 1) \times \frac{\sigma(p_2^{a_2})}{p_2^2} \times \prod_{i=3}^{m} \frac{\sigma(p_i^{a_i})}{p_i^2} \;=\; \frac{4p_1^2}{p_1 + 1} .$$

However $\sigma(5^5)/5^2 = 2906/25 \leq \sigma(p_2^{a_2})/p_2^2$ in which case $(2^{e+1} - 1) \times \frac{2906}{25} < \frac{4p_1^2}{p_1+1} < 4p_1$; the latter inequality is only satisfied by primes $p_1 \geq 173$. □

Proposition 3. *If $n \in \mathcal{K}$ is a solution with $p_1 \equiv 3 \pmod 8$ such that n is not divisible by the fourth power of any odd prime, then p_1 cannot divide $2^{e+1} - 1$.*

Proof. Using [2, Theorem 3], one can express

$$n \;=\; 2^e \times p_1 \times \prod_{i=2}^{m} p_2^2$$

and moreover $2^{e+1} - 1 \leq 4p_1^2/(p_1 + 1) < 4p_1$. Thus under the assumption that $p_1 \mid 2^{e+1} - 1$, either $p_1 = 2^{e+1} - 1$ or $3p_1 = 2^{e+1} - 1$.

First suppose that $p_1 = 2^{e+1} - 1$. From the proof of [2, Theorem 3], one has

$$\frac{1}{4} \times \prod_{i=2}^{m} \frac{p_i^2 + p_i + 1}{p_i^2} \;\leq\; 0.73$$

consequently $(p_1 - 1) \times 0.73 \geq p_1$. The latter inequality implies $p_1 < 3$, which is false.

Alternatively if $3p_1 = 2^{e+1} - 1$, because $9 \neq 2^{e+1} - 1$ for any value of e, clearly $3 \neq p_1$, so we can instead set $p_2 = 3$. Similarly $13 = 3^3 + 3 + 1 \neq p_1$, and $13^2 + 13 + 1 = 3 \times 61$ with $61 \neq p_1$. However $3 \mid 61^2 + 61 + 1$ giving at least three powers of 3 dividing the left hand side of $\sigma(n) = \gamma(n)^2$, which again yields a contradiction. □

The following three technical lemmas are key ingredients in the proof of Theorem 1.

Lemma 5. *If $n \in \mathcal{K}$ is divisible by 3, there exists an odd prime p such that $p^4 \mid n$.*

Proof. Assume (hypothetically) n is not divisible by the fourth power of an odd prime. If $p_1 \equiv 3 \pmod 8$ then using Lemma 1, we can write

$$(2^{e+1} - 1) \times (p_1 + 1) \times (p_2^2 + p_2 + 1) \times \cdots \times (p_m^2 + p_m + 1) \;=\; 4p_1^2 p_2^2 \cdots p_m^2.$$

By Lemma 1 once more, we know $p_1 \neq 3$ so instead put $p_2 = 3$. Consider the system:

$$\begin{array}{rcllll} 3^2 + 3 + 1 &=& 13; & 13 \equiv 5 \bmod 8, & 13 \neq p_1, & 13 = p_3 \\ 13^2 + 13 + 1 &=& 3 \times 61; & 61 \equiv 5 \bmod 8, & 61 \neq p_1, & p_4 = 61 \\ 61^2 + 61 + 1 &=& 3 \times 13 \times 97; & 97 \equiv 1 \bmod 8, & 97 \neq p_1, & p_5 = 97. \end{array}$$

We observe that the left hand side of the previous equation must be divisible by $3^3 = 27$ whilst the right hand side is only divisible by $3^2 = 9$, yielding a contradiction.

If $p_1 \equiv 1 \pmod 4$, one has the decomposition

$$(2^{e+1} - 1) \times (p_1 + 1) \times (p_2 + 1) \times (p_3^2 + p_3 + 1) \times \cdots \times (p_m^2 + p_m + 1) \;=\; 4p_1^2 p_2^2 \cdots p_m^2.$$

Neither p_1 nor p_2 can be 3, thus we may take $p_3 = 3$.

If $p_1 = 13$ then $p_1 + 1 = 2 \times 7$, and we set $p_4 = 7$; therefore $7^2 + 7 + 1 = 3 \times 19$ and $19^2 + 19 + 1 = 3 \times 127$, again giving too many 3's.

If neither p_1 nor p_2 is 13, we can choose $p_4 = 13$ and thereby obtain $13^2 + 13 + 1 = 3 \times 61$.

If $61 = p_1$ or p_2 (let's say $p_1 = 61$), we can write $n = 2^e \cdot 61 \cdot p_2 \cdot p_3^2 \cdots p_m^2$ and so $p_1 + 1 = 2 \times 31$ with $31 \neq p_2$. Consequently we can choose $p_4 = 31$, leading to the equation $\sigma(31^2) = 31^2 + 31 + 1 = 3 \times 331$ and again too many 3's.

Finally if $61 \neq p_1, p_2$ then we still pick up an additional 3, since $3 \mid 61^2 + 61 + 1$. □

Lemma 6. *If a solution $n \in \mathcal{K}$ is not divisible by the fourth power of an odd prime and $p_1 \equiv 3 \pmod 8$, then $3 \mid n$.*

Proof. Suppose $n \in \mathcal{K}$ but $3 \nmid n$. In general, if a prime $q \mid p^2 + p + 1$ then either $q = 3$, or we must have $q \equiv 1 \pmod 3$ so that $3 \mid q^2 + q + 1$. Now from the expression

$$(2^{e+1} - 1) \times (p_1 + 1) \times (p_2^2 + p_2 + 1) \times \cdots \times (p_m^2 + p_m + 1) \;=\; 4p_1^2 p_2^2 \cdots p_m^2$$

we can define $Q := \prod_{i=2}^{m}(p_i^2 + p_i + 1)$. Because $3 \nmid n$, each prime number p_j with $1 \leq j \leq m$ which appears as a factor of Q does not appear in the form $p_i^2 + p_i + 1$; this means we must have $Q \mid p_1^2$. However by Lemma 2, the integer Q has at least three quadratic factors, giving rise to a contradiction. □

Lemma 7. *If $n \in \mathcal{K}$ satisfies $p_1 \equiv 1 \pmod 4$ and $3 \nmid n$, then n is divisible by the fourth power of an odd prime.*

Proof. Suppose $p_1 \equiv p_2 \equiv 1 \pmod 4$. Then in the notation of Lemma 6, it follows that there are two quadratic factors for $Q = \prod_{i=2}^{m}(p_i^2 + p_i + 1)$ and (following cancelation) three possible forms for the equation $\sigma(n) = \gamma(n)^2$. We shall treat each of these separately.

Case (1):

$$\begin{aligned} p_3^2 + p_3 + 1 &= p_1 \\ p_4^2 + p_4 + 1 &= p_2 \\ (2^{e+1} - 1)\left(\frac{p_1 + 1}{2}\right)\left(\frac{p_2 + 1}{2}\right) &= p_1 p_2 p_3^2 p_4^2. \end{aligned}$$

Note that $\frac{p_2+1}{2}$ has at least one prime divisor, and at most three prime divisors.

(1.1) If $\frac{p_2+1}{2}$ has only one prime divisor then $\frac{p_2+1}{2} = p_1$; under this scenario, there are seven possibilities for $\frac{p_1+1}{2}$.

(1.1.1) If $\frac{p_1+1}{2} = p_2$ then $2^{e+1} - 1 = p_3^2 p_4^2$, which is impossible.

(1.1.2) If $\frac{p_1+1}{2} = p_3^2$ then $p_3 \mid p_1 + 1$; however $p_3 \mid p_1 - 1$ so $p_3 \mid \gcd(p_1 + 1, p_1 - 1) = 2$, which is impossible.

(1.1.3) If $\frac{p_1+1}{2} = p_4^2$ then

$$p_3^2 + p_3 + 1 \;=\; \frac{p_2 + 1}{2} \;=\; \frac{p_4^2 + p_4 + 2}{2}$$

implying both $p_3 \mid p_4 + 1$ and $p_4 \mid p_3 + 1$, which is clearly false.

(1.1.4) If $\frac{p_1+1}{2} = p_3 p_4$ then $p_3 \mid p_1 + 1$; however $p_3 \mid p_1 - 1$ hence $p_3 \mid (p_1 + 1, p_1 - 1) = 2$, which is again false.

(1.1.5) If $\frac{p_1+1}{2} = p_2 p_3^2$ then $2^{e+1} - 1 = p_4^2$, which is impossible.

(1.1.6) If $\frac{p_1+1}{2} = p_2 p_4^2$ then $2^{e+1} - 1 = p_3^2$, which is impossible.

(1.1.7) If $\frac{p_1+1}{2} = p_2 p_3 p_4$ then $p_3 \mid p_1 + 1$; now $p_3 \mid p_1 - 1$ thus $p_3 \mid \gcd(p_1 + 1, p_1 - 1) = 2$, which is false.

(1.2) If $\frac{p_2+1}{2}$ has two prime divisors, either $\frac{p_2+1}{2} = p_3^2$; or p_4^2; or p_3p_4.

(1.2.1) If $\frac{p_2+1}{2} = p_3^2$, then either $\frac{p_1+1}{2} = p_2$ or $\frac{p_1+1}{2} = p_4^2$:

(1.2.1.1) If $\frac{p_1+1}{2} = p_2$ then one has $2p_4(p_4+1) = p_3(p_3+1)$, which implies $p_3 \mid p_4+1$ and $p_4 \mid p_3+1$; the last two conditions are incompatible.

(1.2.1.2) If $\frac{p_1+1}{2} = p_4^2$ then $p_4(p_4+1) = 2(p_3+1)(p_3-1)$, which implies that $p_4 < p_3$; further $p_3(p_3+1) = 2(p_4+1)(p_4-1)$ which implies $p_3 < p_4$, impossible!

(1.2.2) If $\frac{p_2+1}{2} = p_4^2$, then either $\frac{p_1+1}{2} = p_2$ or $\frac{p_1+1}{2} = p_3^2$:

(1.2.2.1) If $\frac{p_1+1}{2} = p_2$ then $p_4 = 2$, which is false.

(1.2.2.2) If $\frac{p_1+1}{2} = p_3^2$ then $p_3 = 2$, which is false.

(1.2.3) If $\frac{p_2+1}{2} = p_3p_4$, then either $\frac{p_1+1}{2} = p_2$ or $\frac{p_1+1}{2} = p_3p_4$:

(1.2.3.1) If $\frac{p_1+1}{2} = p_2$ then $p_3 \mid p_4+1$ and $p_4 \mid p_3+1$, which is impossible.

(1.2.3.2) If $\frac{p_1+1}{2} = p_3p_4$ then $p_1 = p_2$, which is false as they are distinct primes.

(1.3) If $\frac{p_2+1}{2}$ has three prime divisors, either $\frac{p_2+1}{2} = p_1p_3^2$; or $p_1p_3p_4$; or $p_1p_4^2$.

(1.3.1) If $\frac{p_2+1}{2} = p_1p_3^2$ then one deduces $2^{e+1}-1 = p_4^2$, which is false.

(1.3.2) If $\frac{p_2+1}{2} = p_1p_3p_4$ then $\frac{p_1+1}{2} = p_2$, which implies that $p_4 \mid p_3+1$ and $p_3 \mid p_4+1$; the latter conditions are incompatible.

(1.3.3) If $\frac{p_2+1}{2} = p_1p_4^2$ then we find $2^{e+1}-1 = p_3^2$, which is false.

Combining (1.1), (1.2), and (1.3) together, clearly Case (1) is impossible in its entirety.

Case (2):

$$\begin{aligned} p_3^2 + p_3 + 1 &= p_1 \\ p_4^2 + p_4 + 1 &= p_1p_2^2 \\ (2^{e+1}-1)\left(\frac{p_1+1}{2}\right)\left(\frac{p_2+1}{2}\right) &= p_3^2p_4^2. \end{aligned}$$

Here $p_3 \equiv p_4 \equiv 2 \pmod 3$, $\frac{p_1+1}{2} \equiv \frac{p_2+1}{2} \equiv 1 \pmod 3$, and there are at least two prime factors in $2^{e+1}-1$ (which being congruent to 3 modulo 4 cannot include p_2, and being congruent to 1 modulo 3 cannot include p_3 or p_4). It follows that there is at least one prime factor in $\frac{p_1+1}{2}$ and $\frac{p_2+1}{2}$ respectively, which leaves us only $\frac{p_1+1}{2} = p_3$ or $\frac{p_1+1}{2} = p_4$, and these are both impossible.

Case (3):

$$\begin{aligned} p_3^2 + p_3 + 1 &= p_1 \\ p_4^2 + p_4 + 1 &= p_1p_2 \\ (2^{e+1}-1)\left(\frac{p_1+1}{2}\right)\left(\frac{p_2+1}{2}\right) &= p_2p_3^2p_4^2. \end{aligned}$$

Note that it cannot happen that one of p_2, p_3, p_4 is the only prime divisor of $\frac{p_2+1}{2}$. Furthermore $2^{e+1}-1$ must have at least two prime divisors, and it cannot be a square; in addition $2^{e+1}-1 \equiv p_2 \equiv \frac{p_1+1}{2} \equiv \frac{p_2+1}{2} \equiv 1 \pmod 3$ and $p_3 \equiv p_4 \equiv 2 \pmod 3$. One therefore deduces

$$\begin{aligned} \frac{p_1+1}{2} &= p_2 \\ 2^{e+1}-1 &= p_3 p_4 \\ \frac{p_2+1}{2} &= p_3 p_4. \end{aligned}$$

From these three equations, we obtain

$$p_3 = \frac{\sqrt{2^{e+5}-31}-1}{2}$$

and by the result of Apéry in Lemma 4, this is clearly an impossible occurrence. □

We are now ready to give the main result of this section.

Theorem 1. *If $n \in \mathcal{K}$ then n is divisible by the fourth power of an* **odd** *prime.*

Proof. Firstly applying Lemma 5, if $n \in \mathcal{K}$ and $3 \mid n$ then $p^4 \mid n$ for some odd prime p. Without loss of generality, we may therefore assume $n \in \mathcal{K}$ and $3 \nmid n$.

If $p_1 \equiv 1 \pmod 4$ then the result is covered by Lemma 7. Likewise if $p_1 \equiv 3 \pmod 8$ then the result is covered by Lemma 6. Finally the remaining case $p_1 \equiv 7 \pmod 8$ is already excluded courtesy of Lemma 1. □

Corollary 1. *If $\{p_1, p_2\} = \{5, 13\}$ then $a_2 \geq 5$.*

Proof. Assume that $a_2 = 1$. Since $\sigma(n) = \gamma(n)^2$, setting $p_3 = 3$ and $p_4 = 7$ implies

$$(2^{e+1}-1)\cdot(2\times 3)\cdot(2\times 7)\times \prod_{i=3}^{m} \sigma(p_i^{a_i}) \;=\; 4\cdot 3^2\cdot 5^2\cdot 7^2\cdot 13^2 \times \prod_{i=5}^{m} p_i^2.$$

Using the divisibility of n by the fourth power of an odd prime (which minimally is 3):

$$(2^{e+1}-1)\times 3\cdot 7\cdot \frac{121}{9}\cdot\frac{31}{5^2}\cdot\frac{57}{7^2}\cdot\frac{183}{13^2} \;<\; 1$$

so $(2^{e+1}-1)\cdot 2 < 1$, which is false for $e \in \mathbb{N}$. Therefore $a_2 > 1$, in which case $a_2 \geq 5$. □

4 The exponents for members of $\mathcal{K}$

We now study the exponents a_i occurring in the decomposition of a De Koninck number. The first step is to adapt an idea of Chen and Chen [3], in order to relate $\omega(n)$ with $\sum_{i=0}^{m} d(a_i+1)$, where $d(x)$ is defined to be the number of divisors of an integer $x \geq 1$. The second step is to apply the AM/GM inequality, then further analyse the exponents.

Lemma 8. *Let a solution $n \in \mathcal{K}$ be represented as the product $n = 2^e \times p_1 \times \prod_{i=2}^{m} p_i^{a_i}$. If we set $p_0 = 2,\ a_0 = e$ and $a_1 = 1$, then there are inequalities*

$$2\omega(n) \ \leq \ \sum_{i=0}^{m} d(a_i+1) \ \leq \ 3\omega(n).$$

Proof. One need only derive the upper bound, since the lower bound follows from (1).

First consider the case where $i \geq 2$ and a_i is even, so p_i is odd. Put $w_i = d(a_i+1)-1$ and write $n_{i,1}, \ldots, n_{i,w_i}$ to denote all the positive integer divisors of a_i+1 other than 1. Let $q_{i,j}$ be a primitive prime divisor of $(p_i^{n_{i,j}}-1)/(p_i-1)$ for $0 \leq i \leq m$ and $1 \leq j \leq w_i$. In particular, there are divisibilities

$$q_{i,j} \ \Big| \ \frac{p_i^{n_{i,j}}-1}{p_i-1} \ \Big| \ \frac{p_i^{a_i+1}-1}{p_i-1}$$

and if $\Omega(x)$ counts the number of prime factors of x with multiplicity, then

$$w_i \ \leq \ \omega\big(\sigma(p_i^{a_i})\big) \ \leq \ \Omega\big(\sigma(p_i^{a_i})\big).$$

Alternatively, if $i = 0$, then primitive divisors exist except for $e+1 = 6$, and in that case

$$w_0 \ = \ d(e+1)-1 \ = \ 3 \ = \ \Omega(2^6-1).$$

Lastly if $i = 1$ or $a_2 = 1$, then we have $1 = d(a_i+1)-1 < 2 \leq \Omega(p_i+1) = \Omega\big(\sigma(p_i^{a_i})\big)$.

Therefore in all cases $d(a_i+1)-1 \leq \Omega\big(\sigma(p_i^{a_i})\big)$, hence there is an inequality

$$\begin{aligned}\sum_{i=0}^{m} d(a_i+1) \ - \ \omega(n) \ &= \ \sum_{i=0}^{m} \big(d(a_i+1)-1\big) \\ &\leq \ \sum_{i=0}^{m} \Omega\big(\sigma(p_i^{a_i})\big) \ = \ \Omega\big(\sigma(n)\big) \ = \ \Omega\big(\gamma(n)^2\big) \ = \ 2\omega(n)\end{aligned}$$

thereby completing the derivation of the upper bound. □

Corollary 2. *If $n \in \mathcal{K}$ then in the notation of Lemma 8, a proportion of more than 2/5 of the numbers $a_i + 1$ must be either prime or prime powers.*

Proof. Because $2^{\omega(a_i+1)} \leq d(a_i+1)$, using the arithmetic-geometric mean and Lemma 8:

$$\left(2^{\sum_{i=0}^{m} \omega(a_i+1)}\right)^{\frac{1}{\omega(n)}} \ \leq\ \frac{\sum_{i=0}^{m} 2^{\omega(a_i+1)}}{\omega(m)} \ \leq\ 3.$$

Moreover, taking the logarithm of both sides, one deduces

$$\sum_{i=0}^{m} \omega(a_i+1) \ \leq\ \left(\frac{\log 3}{\log 2}\right) \times \omega(n).$$

For an integer $i \geq 1$, let $n_i := \#\{j : \omega(a_j+1) = i\}$. Then the above inequality becomes

$$n_1 + 2n_2 + 3n_3 + \cdots \ \leq\ \left(\frac{\log 3}{\log 2}\right) \times (n_1 + n_2 + \cdots)$$

which implies

$$\left(2 - \frac{\log 3}{\log 2}\right)(n_2 + n_3 + \cdots) \ \leq\ \left(2 - \frac{\log 3}{\log 2}\right) n_2 + \left(3 - \frac{\log 3}{\log 2}\right) n_3 + \cdots \ \leq\ \left(\frac{\log 3}{\log 2} - 1\right) n_1.$$

Rearranging the n_i's yields

$$n_1 + n_2 + \cdots \ \leq\ \left(\frac{\frac{\log 3}{\log 2} - 1}{2 - \frac{\log 3}{\log 2}} + 1\right) \times n_1$$

and as the bracketed term equals 2.41 to two decimal places, we conclude that

$$\frac{2}{5} \ <\ \frac{1}{2.41} \ \leq\ \frac{n_1}{n_1 + n_2 + \cdots + n_m} \ \leq\ n_1$$

as required. □

5 Counting the elements in $\mathcal{K} \cap [1, x]$

For every real $x > 0$, we will from now on use the notation $\mathcal{K}(x) := \mathcal{K} \cap [1, x]$. In [2, Theorem 4], it was shown that the size of the solutions $\mathcal{K}(x)$ is asymptotically bounded by $x^{1/4+o(1)}$ as x tends to infinity (and this result was itself an improvement on the work of Pomerance and Pollack [8], which instead gave an upper bound of $x^{1/3+o(1)}$). In this section we will sharpen the bound still further, as described directly below.

Theorem 2. *The estimate*

$$\#\mathcal{K}(x) \ \leq \ x^{1/6+o(1)}$$

holds as $x \to \infty$.

Proof. Let $n > 1$ be in $\mathcal{K}(x)$, so we may express it as $n = A \times B$ where $\gcd(A, B) = 1$, with A squarefree and B squarefull. Exploiting Lemma 1, then $A \in \{p_1, 2p_1, p_1p_2, 2p_1p_2\}$ and B is divisible by at least one prime to the fourth power or greater.

Under the notation of Lemma 3, one can write

$$\frac{N}{D} \ = \ \frac{\sigma(n)}{n} \ = \ \frac{\gamma(n)^2}{n} \ = \ \frac{\gamma(A)^2}{A} \times \frac{\gamma(B)^2}{B} \ = \ \frac{A}{B/\gamma(B)^2} \ > \ 1$$

with $\gcd\big(A, B/\gamma(B)^2\big) = 1$. It follows that $B/\gamma(B)^2 < A$, whence

$$\frac{B^2}{\gamma(B)^2} < AB = n \leq x \quad \Longrightarrow \quad \frac{B}{\gamma(B)} \leq \sqrt{x}.$$

Now by Lemma 1, we can always decompose $B = \delta \times C^2 \times D$ where $\delta \in \{1, 2^3\}$, C is a squarefree integer, D is a 4-full integer, and such that δ, C and D are pairwise coprime. As a consequence,

$$\frac{B}{\gamma(B)} \ = \ \frac{\delta}{\gamma(\delta)} \times C \times \frac{D}{\gamma(D)} \quad \Longrightarrow \quad \frac{D}{\gamma(D)} \ \leq \ \sqrt{x}.$$

In addition

$$\frac{B}{\gamma(B)^2} \ = \ \frac{\delta}{\gamma(\delta)^2} \times \frac{D}{\gamma(D)^2}$$

so that

$$\frac{B}{\gamma(B)^2} \ = \ \frac{D}{\gamma(D)^2} \quad \text{or} \quad \frac{B}{\gamma(B)^2} \ = \ 2 \times \frac{D}{\gamma(D)^2}.$$

Moreover one knows that $D/\gamma(D) \leq \sqrt{x}$ above, which means $D/\gamma(D)^2 \leq \sqrt{x}$.

Now if two 4-full numbers D_1 and D_2 satisfy $D_1/\gamma(D_1) = D_2/\gamma(D_2)$, then we must also have $D_1/\gamma(D_1)^2 = D_2/\gamma(D_2)^2$. Hence the number of choices for $D/\gamma(D)^2 \leq \sqrt{x}$ with $D/\gamma(D) \leq \sqrt{x}$ and D 4-full, is less than or equal to the number of choices for $D/\gamma(D) \leq \sqrt{x}$ which is of type $x^{\frac{1}{6}+o(1)}$.

Therefore the number of choices for $B/\gamma(B)^2$ is also $x^{\frac{1}{6}+o(1)}$, and the proof is completed upon applying Lemma 3. $\square$

6 Applications of the product compactification

For each prime p, let $\mathbb{N}_p$ denote the one point compactification of $\mathbb{N}$; in particular, each finite point $n \in \mathbb{N}$ is itself an open set, and a basis for the neighborhoods of the point at infinity, p^∞ say, is given by the open sets $U_p^{(\epsilon)} = \left\{p^e \in \mathbb{N} : e \geq 1/\epsilon\right\} \cup \{p^\infty\}$ with $\epsilon > 0$. If $\mathbb{P}$ indicates the set of prime numbers, let us write

$$\hat{\mathbb{N}} := \prod_{p \in \mathbb{P}} \mathbb{N}_p$$

for the product of these indexed spaces, endowed with the standard product topology. Then $\hat{\mathbb{N}}$ is a compact metrizable space so it is sequentially compact, hence every sequence in $\hat{\mathbb{N}}$ has a convergent subsequence.

Remark: We shall call $\hat{\mathbb{N}}$ equipped with its topology the *product compactification of* $\mathbb{N}$. A nice account detailing properties of the so-called 'supernatural topology' in attacking the odd perfect number problem, is given by Pollack in [7].

Consider now the more general equation

$$\sigma(n)^\alpha \times \phi(n)^\beta = \theta \times n^\mu \times \gamma(n)^\tau \tag{3}$$

where $\alpha, \beta, \mu, \tau \in \mathbb{Z}$ and $\theta > 0$ is a rational number. Write $\mathcal{K} = \mathcal{K}_{\alpha,\beta,\mu,\tau}$ for the set of solutions

$$\mathcal{K}_{\alpha,\beta,\mu,\tau} = \left\{n \in \mathbb{N} : \sigma(n)^\alpha \times \phi(n)^\beta = \theta \times n^\mu \times \gamma(n)^\tau\right\}$$

which clearly depends on the initial choice of quintuple $(\alpha, \beta, \mu, \tau, \theta)$.

Theorem 3. *Let $\mathcal{P} \subset \mathbb{P}$ denote a fixed finite set of primes, and assume that $\alpha + \beta > \mu$. Then there exist only finitely many $n \in \mathcal{K}$ with support in $\mathcal{P}$.*

Before we give the demonstration, we point out that choosing $\alpha = 1, \beta = 0, \mu = 0, \tau = 2$ and $\theta = 1$ implies there exist only finitely many solutions to De Koninck's equation (1), supported on any prescribed finite set of primes $\mathcal{P}$.

Proof. Given $A, B, M, T \in \mathbb{Z}$, define a multiplicative function $h = h_{A,B,M,T} : \mathbb{N} \to \mathbb{Q}_{>0}$ by the formula

$$h(n) := \frac{\sigma(n)^A \times \phi(n)^B}{n^M \times \gamma(n)^T}.$$

For every $r \geq 1$ and at each prime p, one calculates that

$$h(p^r) = p^{A-B-T}(p-1)^{B-A}(1 - p^{-r-1})^A \times (p^r)^{A+B-M}$$

while $h(1) = 1$. This naturally leads us to the definition

$$\hat{h}(p^{\infty}) \;:=\; \begin{cases} \infty & \text{if } A+B > M \\ 0 & \text{if } A+B < M \\ p^{A-B-T}(p-1)^{B-A} & \text{if } A+B = M, \end{cases}$$

and provides a unique extension $\hat{h} : \hat{\mathbb{N}} \to \mathbb{R} \cup \{\infty\}$ of the original arithmetic function h. In fact if $A+B = M$ and $T = 0$, one can then show $\hat{h}$ is continuous on the monoid $\hat{\mathbb{N}}$.

Fix a finite set of primes $\mathcal{L} = \{l_1, \ldots, l_k\}$, and put

$$\mathbb{N}_{\mathcal{L}} \;:=\; \big\{ n \in \mathbb{N} : n = l_1^{e_1} \cdots l_k^{e_k},\ e_j \geq 1 \big\}.$$

Key Claim: If $A+B \geq M$ then $h\big|_{\mathbb{N}_{\mathcal{L}}}$ is monotonic increasing with respect to divisibility.

To establish this claim suppose that $n = n' \times l_j^{e_j}$ with $n' \in \mathbb{N}_{\mathcal{L}\setminus\{l_j\}}$, and set $m = n' \times l_j^{e_j+1}$. Then $h(m) = h(n') \times h(l_j^{e_j+1})$ and

$$\begin{aligned} h\big(l_j^{e_j+1}\big) \;&=\; \frac{\sigma(l_j^{e_j+1})^A \times \phi(l_j^{e_j+1})^B}{l_j^{(e_j+1)M} \times \gamma(l_j^{e_j+1})^T} \\ &=\; \left(\frac{\sigma(l_j^{e_j+1})}{\sigma(l_j^{e_j})}\right)^A \times l_j^{B-M} \times \frac{\sigma(l_j^{e_j})^A \phi(l_j^{e_j})^B}{l_j^{e_j M}\gamma(l_j^{e_j})^T} \;=\; l_j^{B-M} \times \left(\frac{l_j^{e_j+2}-1}{l_j^{e_j+1}-1}\right)^A \times h\big(l_j^{e_j}\big). \end{aligned}$$

However

$$\begin{aligned} l_j^{B-M} \times \left(\frac{l_j^{e_j+2}-1}{l_j^{e_j+1}-1}\right)^A \;&=\; l_j^{B-M} \times \left(\frac{l_j^{e_j+2}-l_j}{l_j^{e_j+1}-1} + \frac{l_j-1}{l_j^{e_j+1}-1}\right)^A \\ &=\; l_j^{B-M} \times \left(l_j \;+\; \frac{l_j-1}{l_j^{e_j+1}-1}\right)^A \;>\; l_j^{A+B-M} \;\geq\; 1 \end{aligned}$$

since $A+B \geq M$. It follows that $h\big(l_j^{e_j+1}\big) > h\big(l_j^{e_j}\big)$, in which case

$$h(m) \;=\; h(n') \times h\big(l_j^{e_j+1}\big) \;>\; h(n') \times h\big(l_j^{e_j}\big) \;=\; h(n).$$

The proof of the claim then follows by induction on the number of primes (with multiplicity) which divide the quotient of a general pair n and m, with $n \mid m$.

Now let us take $A = \alpha$, and choose $B, M \in \mathbb{Z}$ such that

$$\mu - \beta \;<\; M - B \;\leq\; \alpha.$$

Suppose there exists a sequence of elements in $\mathcal{K}$ supported on $\mathcal{P}$ which are all distinct. Under the supernatural topology, there exists a subsequence $(N_i)_{i\geq 1}$ and a limit $N_o \in \hat{\mathbb{N}}$ such that $N_i \to N_o$. The element N_o is supported on $\mathcal{P}$, otherwise at least one of the N_i would also not be supported on $\mathcal{P}$. We may therefore write $N_o = A \times B^\infty$ where $\mathrm{supp}(A) \subset \mathcal{P}$, $\mathrm{supp}(B) \subset \mathcal{P}$, and $\gcd(A, B) = 1$ with B squarefree. Furthermore

$$\mathrm{supp}(A) \cup \mathrm{supp}(B) \;=\; \mathcal{L} \;=\; \{l_1, \ldots, l_k\}, \text{ say.}$$

Then there exists a subsequence $(N_{i_j})_{j\geq 1}$ of the sequence $(N_i)_{i\geq 1}$ satisfying for all $j \geq 1$:

(i) $\mathrm{supp}(N_{i_j}) = \mathcal{L}$,

(ii) N_{i_j} properly divides $N_{i_{j+1}}$, and

(iii) $A \| N_{i_j}$.

Each $N_{i_j} \in \mathcal{K}$ and h is monotonic on the monoid $(\mathbb{N}, \times)$, hence for all $j \geq 2$ one has

$$\begin{aligned} 0 \;<\; h(N_{i_1}) \;<\; h(N_{i_j}) &\overset{\text{by (3)}}{=} \frac{\theta \times \phi(N_{i_j})^{B-\beta}}{N_{i_j}^{M-\mu} \times \gamma(N_{i_j})^{T-\tau}} \\ &= \frac{\phi(N_{i_j})^{B-\beta}}{N_{i_j}^{M-\mu}} \times \frac{\theta}{(\prod_{s=1}^k l_s)^{T-\tau}} \\ &\leq N_{i_j}^{(B-\beta)-(M-\mu)} \times \frac{\theta}{(\prod_{s=1}^k l_s)^{T-\tau}} \end{aligned}$$

which tends to zero as $j \to \infty$ since $M - B > \mu - \beta$.

This immediately yields a contradiction, and completes the proof of the theorem. □

7 Final Comments

In Theorem 2, we believe it should be possible to reduce the upper bound to $x^{o(1)}$. Moreover extending the list of bad De Koninck primes, for example by finding additional infinite sets, seems readily achievable.

In the fundamental Lemma 1, showing that the exponent e of the power of 2 equals 1 (or at least is odd) looks like a reasonable goal, but we have been unable to prove this.

Lastly, extending the method of Theorem 3 to include subsets of $\mathcal{K}$ with prime support of bounded size, seems altogether more challenging.

References

[1] R. Apéry, *Sur une équation diophantienne*, C. R. Acad. Sci. Paris Sér. A **251** (1960), 1263–1264, and 1451–1452.

[2] K. A. Broughan, J.M. De Koninck, I. Kátai and F. Luca, *On integers for which the sum of divisors is the square of the squarefree core*, Journal of Integer Sequences, **15** (2012), 1–12.

[3] F. J. Chen and Y. G. Chen, *On odd perfect numbers*, Bull. Aust. Math. Soc. **86** (2012), 510–514.

[4] R. K. Guy, *Unsolved Problems in Number Theory, Third Edition*, Springer, 2004.

[5] F. Luca, *On numbers n for which the prime factors of $\sigma(n)$ are among the prime factors of n*, Result. Math., **45** (2004), 79–87.

[6] P. Pollack, *The greatest common divisor of a number and its sum of divisors*, Michigan Math. J. **60** (2011), 199–214.

[7] P. Pollack, *Finiteness theorems for perfect numbers and their kin*, American Math. Monthly **119** (2012), 670–681.

[8] P. Pollack and C. Pomerance, *Prime–perfect numbers*, Integers **12** no. 6 (2009), 1417–1437.