

QUALIFIED DIFFERENCE SETS FROM UNIONS OF CYCLOTOMIC CLASSES

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ABSTRACT. Qualified difference sets (QDS) composed of unions of cyclotomic classes are discussed. An exhaustive computer search for such QDS and modified QDS that also possess the zero residue has been conducted for all powers $n = 4, 6, 8$ and 10 . Two new families were discovered in the case $n = 8$ and some new isolated systems were discovered for $n = 6$ and $n = 10$.

1. INTRODUCTION

We define a qualified difference set (QDS) as follows.

Definition 1.1. *Let $R = \{r_1, r_2, r_3, \dots, r_k\}$ be a k -element set of distinct non-zero residues modulo an integer v . We call R a qualified difference set (QDS) if there exists some non-zero integer $m \notin R$ which is such that if we form all the non-zero differences*

$$(1.1) \quad r_i - mr_j \pmod{v} \quad 1 \leq i, j \leq k$$

we obtain every positive integer $\leq v-1$ exactly λ times. We call m a qualifier of multiplicity λ for the set R .

If zero is counted as a residue, then further such sets are possible. These are called *modified qualified difference sets*, (MQDS) by virtue of the modification introduced by the inclusion of the zero element. We define these sets as follows.

Definition 1.2. *Let $R^* = \{r_0, r_1, r_2, \dots, r_k\}$ be a $k + 1$ -element set of residues of an integer v , where $r_0 = 0$. We call R^* a modified qualified*

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difference set (MQDS) if there exists some non-zero integer $m \notin R^$ which is such that if we form all the differences*

$$(1.2) \quad r_i - mr_j \pmod{v} \quad 0 \leq i, j \leq k$$

we obtain every positive integer $\leq v - 1$ exactly λ times and zero exactly once. We call m a qualifier of multiplicity λ for the set R^ .*

When QDS and MQDS are generated from n th power residues of a prime $v = p = nk + 1$ with n a positive integer, we obtain the special cases of qualified residue difference sets (QRDS) and modified qualified residue difference sets (MQRDS) respectively. These sets were introduced by Jennings and Byard [12, 13].

QDS and MQDS have similar properties to the residue difference sets and modified residue difference sets respectively, which were discussed in detail in 1953 by Lehmer [15]. All four classes of set possess similarly attractive properties. In particular when mapped onto a binary $(0, 1)$ array, they possess a two-valued correlation function (see equations (6.1) - (6.3) in the appendix). This property suggests potential applications in areas such as image formation [4, 7, 19], signal processing [17] and aperture synthesis [14].

In his subsequent extensive survey, Hall [9] extended the notion of ‘residue difference sets’ and discovered a new family of difference set that can be created from a union of 6th power *cyclotomic classes*, where we define the n th power cyclotomic class $C(c_i)$ for the prime p , by

$$(1.3) \quad C(c_i) \equiv \{g^{un+c_i} \pmod{p} : 0 \leq u \leq f - 1\}$$

where $p = nf + 1$ and g is a primitive root of p . Hall discovered that the union of those 6th power cyclotomic classes with indices congruent to 0, 1 and 3 modulo 6 form a difference set when p is of the form $p = 4\alpha^2 + 27$ (integer α), for an appropriate choice of primitive root g , such that $\text{ind}_g 3 = 1$. In this paper we demonstrate the existence of QDS and MQDS that are similarly composed of unions of cyclotomic classes. These sets are defined as follows.

Definition 1.3. Let $p = nf + 1$ be a prime and $k = tf$. Let $C = \{c_1, c_2, \dots, c_t\}$ be a set of t residue classes that are all distinct modulo n . Now let R be the k -element set consisting of the union of n th power cyclotomic classes derived from the set C as follows:

$$(1.4) \quad R = \{r_i : 1 \leq i \leq k\} = C(c_1) \cup C(c_2) \cup \dots \cup C(c_t)$$

and let R^* be the $k + 1$ element set, defined by

$$(1.5) \quad R^* = \{r_i : 0 \leq i \leq k\} = R \cup \{0\}, \quad r_0 = 0$$

where $C(c_i)$ is defined by congruence (1.3). Then R is a QDS if, for a suitable choice of qualifier m , the conditions in Definition 1.1 are satisfied, and R^* is a MQDS if, for a suitable choice of qualifier m , the conditions of Definition 1.2 are satisfied.

The purpose of this article is to describe the results of an exhaustive search for QDS and MQDS created from the unions of cyclotomic classes for the cases $n = 4, 6, 8$ and 10 .

2. RESIDUE CLASSES AND CYCLOTOMIC CONSTANTS

Let $p = nf + 1$ be a prime and g a primitive root of p . An integer N is said to be in *residue class i* if the following congruence holds for some integer u :

$$(2.1) \quad N \equiv g^{un+i} \pmod{p}.$$

The *cyclotomic constant (i, j)* denotes the number of solutions to the congruence

$$(2.2) \quad g^{un+i} + 1 \equiv g^{vn+j} \pmod{p}$$

where $0 \leq i, j \leq n - 1$ and $0 \leq u, v \leq f - 1$. The following further results, due to Dickson [8], are required too:

$$(2.3) \quad (i, j) = (i + \gamma_1 n, j + \gamma_2 n)$$

where γ_1 and γ_2 are integers,

$$(2.4) \quad (i, j) = (n - i, j - i) = (-i, j - i) \text{ for all } f$$

$$(2.5) \quad (i, j) = (j, i) \text{ if } f \text{ is even.}$$

3. THEORY

In order to investigate whether QDS and MQDS can be generated from unions of cyclotomic classes, it is necessary to determine how often the non-zero differences occur between the elements of sets of different cyclotomic classes. This can be done as follows, using an approach similar to that adopted by Hall [9] and a further search for residue difference sets of order $n = 10$ conducted by Hayashi [11]. Following (1.1), consider the congruence

$$(3.1) \quad Y - mX \equiv d \pmod{p}$$

where Y is in residue class j , X is in residue class i , m is in residue class $\sigma \neq 0$ and d is in residue class s . Using these conditions, we get:

$$g^{An+j} - g^{Bn+i+\sigma} \equiv g^{\kappa n+s} \pmod{p}$$

where A , B and κ are integers. Multiplying this congruence by $g^{-\kappa n-s}$ and rearranging gives

$$(3.2) \quad g^{B'n+i+\sigma-s} + 1 \equiv g^{A'n+j-s} \pmod{p}$$

where A' and B' are also integers. Now by the cyclotomic constant equation (2.2), we see that congruence (3.2), and hence congruence (3.1), has $(i + \sigma - s, j - s)$ solutions. This enables us to calculate how often each difference d arises from congruence (3.1), which is derived from different n th power cyclotomic classes. For the set of residue classes $C = \{c_1, c_2, \dots, c_t\}$, the general equation for the number $N(s)$ of times the difference d in residue class s occurs is given by

$$(3.3) \quad N(s) = \sum_{i \in C} \sum_{j \in C} (i + \sigma - s, j - s).$$

Note by equation (2.3) that $N(s + \gamma n) = N(s)$ for integer γ . As there are $n - 1$ residue classes, then there are at most $n - 1$ different values for $N(s)$, although in practice some of the values of $N(s)$ will repeat for different values of s . If conditions can be determined that make each value of $N(s)$ equal, then a QDS will result. If no such conditions exist, there will be no corresponding QDS.

In the case of a MQDS, the residue 0 is added to the union of cyclotomic classes. Here we denote the set of residue classes by $C = \{c_1, c_2, \dots, c_t\}_0$. In this case we need to consider the two additional differences:

$$(3.4) \quad Y - 0 \equiv d \pmod{p}$$

$$(3.5) \quad 0 - mX \equiv d \pmod{p}.$$

For congruence (3.4) we have $Y \equiv d$ and so d , which is in residue class s , also is in residue class j . Therefore $j = s$ and hence $s \in C$. This means that we need to add 1 to the summation $N(s)$ in equation (3.3) if $s \in C$. Congruence (3.5) is a little more complicated. Here we have $(-1)(mX) \equiv d$ and mX is in residue class $\sigma + i$. Now

$$(3.6) \quad (-1) \equiv g^{(p-1)/2} \equiv g^{nf/2} \equiv \begin{cases} g^{n(f/2)} & \text{if } f \text{ is even} \\ g^{n(f-1)/2+n/2} & \text{if } f \text{ is odd.} \end{cases}$$

Therefore if f is even, then -1 is in residue class 0. In this case, by equation (3.5), d , which is in residue class s , is also in residue class $\sigma + i + 0$ (i.e. the residue class of $-mX$). Therefore $i = s - \sigma$ and so we must add 1 to the summation for $N(s)$ in equation (3.3) if $s - \sigma \in C$. If f is odd, then by equation (3.6) -1 is in residue class $n/2$. Here, by equation (3.5), d , which is in residue class s , is also in residue class $\sigma + i + n/2$. Therefore, $i = s - \sigma - n/2$ and so we must add 1 to the summation for $N(s)$ in equation (3.3) if $s - \sigma - n/2 \in C$.

Equation (3.3) has been applied to the cases $n = 4, 6, 8$ and 10 to determine if any QDS or MQDS composed of unions of cyclotomic classes exist.

In each case, an exhaustive computer search has been completed for the range $1 \leq \sigma \leq n - 1$, for f odd and f even (note that $\sigma = 0$ corresponds to difference sets, which is not the subject of this study).

An important type of equivalence between the sets is the concept of ‘complementary’ sets. This is a familiar notion in the study of difference sets (see, for example, Baumert [2, p. 2-3]) and the same applies to QDS and MQDS. If R is a QDS, then the set $R^* = \{Z_p\} - \{R\}$ is a MQDS which is the complement of R , that has the same qualifier m . Stated differently, if C is the set of residue classes that give a QDS, then $C^* = \{Z_n\} - \{C\}$ is the set of residue classes that, with the residue zero, gives the corresponding complementary MQDS. The detailed analysis of this point is given in the Appendix. Because of this equivalence, it is only necessary to analyse cases where $t \leq n/2$.

For each value of n , there are many other equivalent cases that arise due to the following two isomorphisms. Firstly, assume that $C = \{c_1, c_2, \dots, c_t\}$ is a set of residue classes that produces a QDS. Therefore, congruence (3.1) and equation (3.3) lead to values of $N(s)$ that are equal for all s . Now consider the set $C' = \{c_1 + q, c_2 + q, \dots, c_t + q\}$ for some integer q . Using equation (3.3) the summation $N(s)$ for the set C' is as follows:

$$\begin{aligned}
 N(s) &= \sum_{i \in C'} \sum_{j \in C'} (i + \sigma - s, j - s) \\
 &= \sum_{i \in C} \sum_{j \in C} (i + q + \sigma - s, j + q - s) \\
 (3.7) \quad &= \sum_{i \in C} \sum_{j \in C} (i + \sigma - [s - q], j - [s - q]) \\
 &= N(s - q).
 \end{aligned}$$

Now, since each value of $N(s)$ is equal for C then the same values for $N(s - q)$ will result for the set C' . Therefore, if $C = \{c_1, c_2, \dots, c_t\}$ is a set of residue

classes that gives a QDS, then the set $C' = \{c_1 + q, c_2 + q, \dots, c_t + q\}$ will also give an isomorphic QDS. In the case of MQDS, R^* , the same argument applies. If $C = \{c_1, c_2, \dots, c_t\}_0$ is a set of residue classes that produces a MQDS then $C^* = \{Z_n\} - \{C\}$ will give a complementary QDS.

Secondly assume that $C = \{c_1, c_2, \dots, c_t\}$ is a set of residue classes that, using the primitive root g , gives a QDS. Choose one of these classes, say c , and let $C(c)$ be the corresponding cyclotomic class. Therefore, by congruence (1.3) we have

$$(3.8) \quad C(c) \equiv \{g^{un+c} \pmod{p} : 0 \leq u \leq f-1\}.$$

Now, let g_1 be another primitive root of p , where

$$(3.9) \quad g_1^z \equiv g \pmod{p}$$

where z must be prime to $p-1 = nf$. Substituting this into (3.8) gives

$$(3.10) \quad C(c) \equiv \{g_1^{zun+zc}\} \equiv \{g_1^{zc}(g_1^n)^{uz} \pmod{p} : 0 \leq u \leq f-1\}.$$

Now, because the integers $u : 0 \leq u \leq f-1$ form a complete residue system modulo f , then because z is prime to f (which is the case, since z must be prime to nf), the integers $uz : 0 \leq u \leq f-1$ also form the same reduced residue system. Therefore, congruence (3.10) becomes

$$(3.11) \quad C(c) \equiv \{g_1^{zc}(g_1^n)^u\} \equiv \{g_1^{un+zc} \pmod{p} : 0 \leq u \leq f-1\}.$$

Comparing (3.8) with (3.11) shows that a QDS derived from $C = \{c_1, c_2, \dots, c_t\}$ and primitive root g will be the same as that derived from $C = \{zc_1, zc_2, \dots, zc_t\}$ and primitive root g_1 where $g_1^z \equiv g \pmod{p}$. In the case of a MQDS, the same argument evidently applies.

The results below are given up to equivalence, either by complementary sets or isomorphism. A major positive result was revealed in the case $n = 8$, This, therefore, is where we begin.

4. RESULTS FOR $n = 8$

In this section we prove the following theorem:

Theorem 4.1. *Qualified difference sets created from the union of 8th power cyclotomic classes $C = \{0, 1\}$ exist for all primes of the form $p = 64z^4 + 128z^3 + 144z^2 + 80z + 17$ and $p = 64z^4 + 48z^2 + 1$ where z is an integer. All other unions of 8th power cyclotomic classes are isomorphic to either of these cases or to previously known QDS, MQDS or residue difference sets.*

Proof. For the case $n = 8$, we need to consider the additional condition of whether 2 is either a biquadratic (i.e. 4th power) residue or a biquadratic non-residue of p . In order to demonstrate the computational methods used, a detailed analysis follows, for the case when f is even, $t = 2$, $C = \{0, 1\}$, $\sigma = 4$ and 2 is a biquadratic nonresidue of p .

For $C = \{0, 1\}$ and $\sigma = 4$, equation (3.3), along with equations (2.3), (2.4) and (2.5), gives the following condition for the number of differences $N(s)$:

$$(4.1) \quad \begin{aligned} s = 0 : & \quad N(0) = (4, 0) + (4, 1) + (5, 0) + (5, 1) \\ s = 1 : & \quad N(1) = (3, 7) + (3, 0) + (4, 7) + (4, 0) \\ s = 2 : & \quad N(2) = (2, 6) + (2, 7) + (3, 6) + (3, 7) \\ s = 3 : & \quad N(3) = (1, 5) + (1, 6) + (2, 5) + (2, 6). \end{aligned}$$

In this case the values for $s = 4, 5, 6, 7$ repeat those for $s = 0, 1, 2, 3$. Following Dickson's work [8, Theorem 11], Berndt et al. demonstrated that the cyclotomic constants for $n = 8$ are determined uniquely by the quadratic partition

$$(4.2) \quad p = a_4^2 + b_4^2 = a_8^2 + 2b_8^2$$

where, for our case $p = 8f + 1$, we have

$$(4.3) \quad a_4 \equiv -1 \pmod{4}, \quad b_4 \equiv a_4 g^{(p-1)/4} \pmod{p}$$

[3, p. 78], and

$$(4.4) \quad a_8 \equiv -1 \pmod{4}, \quad 2b_8 \equiv a_8(g^f + g^{3f}) \pmod{p}$$

[3, p. 109] where a_4, b_4, a_8 and b_8 are integers in the nomenclature of Berndt et al. The actual cyclotomic constants for $n = 8$ have been calculated, using Dickson's results [8], by Lehmer, who lists them in the appendix of her paper for the cases $p = 16\alpha + 1$ and $p = 16\alpha + 9$ (integer α) [16]. Since we now have $p = 8f + 1$ and f is even, we require her list for $p = 16\alpha + 1$ [16, p. 116]. Substituting these cyclotomic constants for the case when 2 is a biquadratic nonresidue of p into equations (4.1) gives:

$$(4.5) \quad \begin{aligned} s = 0 : \quad & 64N(0) = 4p - 12 + 4a_4 + 4a_8 \\ s = 1 : \quad & 64N(1) = 4p - 12 + 4a_4 + 4a_8 \\ s = 2 : \quad & 64N(2) = 4p + 4 - 4a_4 - 4a_8 + 8b_4 - 16b_8 \\ s = 3 : \quad & 64N(3) = 4p + 4 - 4a_4 - 4a_8 + 16b_8 - 8b_4. \end{aligned}$$

(Note that in her paper, Lehmer instead uses the symbols x, y, a, b , where $x = -a_4$, $a = -a_8$, $2y = b_4$ and $b = b_8$). Now, if all values $N(s)$ are equal then a QDS will result. Putting this restriction on the equations in (4.5) gives:

$$(4.6) \quad \begin{aligned} b_4 &= 2b_8 \\ a_4 + a_8 &= 2. \end{aligned}$$

Combining equations (4.6) and (4.2), and setting $x = -a_4$ and $2y = b_4$ for ease of notation, gives

$$(4.7) \quad \begin{aligned} p &= x^2 + 8x + 8 & x &\equiv 1 \pmod{4} \\ y^2 &= 2 + 2x \end{aligned}$$

and hence $p = 64z^4 + 128z^3 + 144z^2 + 80z + 17$ as in the statement of Theorem 4.1. Now, under the conditions in equation (4.2), 2 is always a biquadratic nonresidue of p of the form in equation (4.7) by the following analysis. Equation (4.7) gives $y^2 = 2(x + 1)$, but since $x \equiv 1 \pmod{4}$ then we have $y^2 = 4(2\eta + 1)$ where $2\eta + 1$ must be an odd square. Substituting

this into equation (4.2) means that $p = x^2 + [4(2\eta + 1)]^2$, and so, since the representation of $p = 8f + 1 \equiv 1 \pmod{4}$ as the sum of two squares is unique up to order and sign, p cannot be represented in the form $p = x^2 + 64\eta_1^2$ for integer η_1 . Therefore 2 must be a biquadratic nonresidue of p by a known theorem on biquadratic reciprocity (see, for example Mollin, [18, Corollary 5.71]) and as a result we will always obtain values for $N(s)$ as given by the equations in (4.5). The sequence of primes of this form starts $p = 17, 433, 2801, 10193, 60017, \dots$

The values of those primitive roots that can be used to generate the QDS are determined by the condition $b_4 = 2b_8$ from equation (4.6). Substituting the values of b_4 and b_8 from equations (4.3) and (4.4) into (4.6) and rearranging gives

$$(4.8) \quad a_8\mu^2 - a_4\mu + a_8 \equiv 0 \pmod{p}$$

where $\mu = g^f$. This quadratic congruence for μ can be demonstrated to give

$$(4.9) \quad (2a_8\mu - a_4)^2 \equiv a_4^2 - 4a_8^2 \pmod{p}.$$

Congruence (4.9) reduces to a simple calculation of the squares $(2a_8\mu - a_4)^2$, which can then be used to calculate the values of $\mu = g^f$ that lead to the condition $b_4 = 2b_8$, and hence those primitive roots g that give QDS for $n = 8$, $\sigma = 4$, $C = \{0, 1\}$ and $p = x^2 + 8x + 8$ ($x \equiv 1 \pmod{4}$). For the converse, assume that we have a prime of the form given in equation (4.7). We can write $p = (x + 2)^2 + 2y^2$ and so, by uniqueness and equation (4.2), $b^2 = y^2$ and so $(x + 2)^2 = a^2$, where $a = -a_8$. Therefore $x + 2 = \pm a$ which, combined with the condition $x \equiv a \equiv 1 \pmod{4}$ means that $x + a = -2$ and so the second of the equations in (4.6) is satisfied. Now since $x + a + 2 = 0$, we have $a = -(x + 2)$ so we can write

$$(4.10) \quad x^2 - 4a^2 = -[x^2 + 2(x^2 + 8x + 8)] = -(x^2 + 2p) \equiv -x^2 \pmod{p}.$$

However, $(-1/p) = 1$ where $(-1/p)$ is the Legendre symbol, since $p \equiv 1 \pmod{4}$. Combining this with (4.10) means that $x^2 - 4a^2$ (i.e. $a_4^2 - 4a_8^2$)

is a square modulo p and so congruence (4.9) always has a solution and so there is always a QDS of the current form under the conditions in (4.7).

A similar analysis for the case when 2 is a biquadratic residue of p leads to another family of QDS. Here we have

$$(4.11) \quad \begin{aligned} b_4 &= 2b_8 \\ a_4 + a_8 &= -2 \end{aligned}$$

which, it can be proved, gives a QDS for all primes of the form

$$(4.12) \quad \begin{aligned} p &= x^2 - 8x + 8 & x &\equiv 1 \pmod{4} \\ y^2 &= 2 - 2x \end{aligned}$$

and hence $p = 64z^4 + 48z^2 + 1$ as in the statement of Theorem 4.1. Because $b_4 = 2b_8$ from (4.11), the primitive root g is calculated exactly in the same manner as for when 2 is a biquadratic nonresidue, using equation (4.9). In this case the sequence of primes starts $p = 113, 1217, 41201, 84673, 644801 \dots$. The proof of Theorem 4.1 is complete.

The well known conjectures of Hardy and Littlewood [10] and their vast generalization by Bateman and Horn [1] can be used to estimate the number of primes of the form

$$p = f(z) = 64z^4 + 128z^3 + 144z^2 + 80z + 17 \text{ or } p = 64z^4 + 48z^2 + 1$$

which occur in the statement of Theorem 4.1 above. In the case of the first polynomial $f(z)$ the conjecture takes the following explicit form:

$$\begin{aligned} Q(x) &= \#\{n \leq x : f(n) \text{ is prime}\}, \\ Q(x) &\sim C_f/4 \int_2^x \frac{dt}{\log t}, \text{ where} \\ C_f &= \prod_{p \equiv 1 \pmod{8}} \left(1 - \frac{1}{p}\right) \left(1 - \frac{4}{p}\right) \cdot \prod_{p \not\equiv 1 \pmod{8}} \left(1 - \frac{1}{p}\right). \end{aligned}$$

An evaluation of $C_f/4$ using 10,000 primes gave a value 1.33792. A more complete set of evaluations is listed in Table 1.

Note that we have used the property, derived heuristically, that $f(z)$ has 4 solutions modulo a prime p if and only if $p \equiv 1 \pmod{8}$.

When we made an explicit count of the number of prime values $p = f(n)$ up to $n = 10^5$ and divided this by the number predicted by Bateman-Horn we obtained the ratio 0.901, showing a degree of feasibility for the conjecture in this case. A set of four evaluations is listed in the Table 1.

| Primes | coefficient | x | actual/Bateman-Horn |
|--------|-------------|---------|---------------------|
| 10,000 | 1.33792 | 99,997 | 0.90138 |
| 11,000 | 1.33866 | 99,998 | 0.90137 |
| 12,000 | 1.33917 | 99,999 | 0.90136 |
| 13,000 | 1.33885 | 100,000 | 0.90135 |

TABLE 1. Evaluation of Bateman-Horn predictions for primes of the form $p = 64z^4 + 128z^3 + 144z^2 + 80z + 17$.

For brevity, the rest of the results for $n = 8$ are given without the detailed computational proofs. As noted in Theorem 4.1 some of these are simply redefinitions of known QDS or MQDS. The list of QDS discovered for $n = 8$ is given in Table 2. For each case in the table, except for $C = \{0, 1\}$, any primitive root g can be used to generate the set.

5. RESULTS FOR $n = 4$, $n = 6$ AND $n = 10$

An exhaustive computer search for QDS created from unions of cyclotomic classes was also completed for $n = 4, 6$ and 10 . The results are discussed in this section. For $n = 4$ and $n = 6$ the following theorem applies:

Theorem 5.1. *A modified difference set created from the unions of 6th power cyclotomic classes $C = \{0, 1\}_0$ exists for $p = 13$. All other unions of 4th power and 6th power cyclotomic classes are isomorphic to previously known QDS, MQDS or residue difference sets.*

The results for the cases $n = 4$ and $n = 6$ are given in Table 3.

In the case $n = 10$ we have the following theorem:

| f | C | p | σ | Comments |
|----------------|--------------------|---|------------|---|
| even | $\{0, 1\}$ | $x^2 + 8x + 8$ $x \equiv 1 \pmod{4}$ $y^2 = 2 + 2x$ | 4 | New family of QDS; g determined by (4.8) |
| even | $\{0, 1\}$ | $x^2 - 8x + 8$ $x \equiv 1 \pmod{4}$ $y^2 = 2 - 2x$ | 4 | New family of QDS; g determined by (4.8) |
| even or odd | $\{0, 4\}$ | $16\alpha^2 + 1$ integer α | 2, 6 | Isomorphic to QRDS with $n = 4$ |
| even or odd | $\{0, 4\}_0$ | $16\alpha^2 + 9$ integer α | 2, 6 | Isomorphic to MQRDS with $n = 4$ |
| even or odd | $\{0, 2, 4, 6\}$ | $8\alpha + 1$ integer α | 1, 3, 5, 7 | Isomorphic to QRDS with $n = 2$ |
| even or odd | $\{0, 2, 4, 6\}_0$ | $8\alpha + 1$ integer α | 1, 3, 5, 7 | Isomorphic to MQRDS with $n = 2$ |

TABLE 2. List of parameters of QDS and MQDS composed of unions of cyclotomic classes for $n = 8$.

Theorem 5.2. *Modified residue difference sets created from the unions of 10th power cyclotomic classes $C = \{0, 2\}_0$ and $C = \{0, 1, 2, 6\}_0$ exist for $p = 41$. All other unions of 10th power cyclotomic classes are isomorphic to previously known QDS, MQDS or residue difference sets.*

The investigation for $n = 10$ revealed similar results to $n = 6$ and are shown in Table 4. In addition to previously known systems, we have two single cases of MQDS for $n = 10$, $p = 41$, $f = 4$ and $\sigma = 5$, namely $t = 2$, $C = \{0, 2\}_0$, and $t = 4$, $C = \{0, 1, 2, 6\}_0$. In the configuration for $t = 4$ any primitive root can be used to generate the MQDS. In the case $t = 2$, however, the choice of primitive root is important. In the case $C = \{0, 2\}_0$ we need to use a primitive root g such that $\text{ind}_g 2 \equiv 1 \text{ or } 4 \pmod{5}$.

| n | f | C | p | σ | Comments |
|-----|----------------|-----------------|------------------------------------|----------|---|
| 4 | even or odd | $\{0, 2\}$ | $4\alpha + 1$ integer α | 1, 3 | Isomorphic to QRDS with $n = 2$ |
| 4 | even or odd | $\{0, 2\}_0$ | $4\alpha + 1$ integer α | 1, 3 | Isomorphic to MQRDS with $n = 2$ |
| 6 | even | $\{0, 1\}_0$ | 13 | 3 | Single case |
| 6 | even | $\{0, 2, 4\}$ | $12\alpha + 1$ integer α | 1, 3, 5 | Isomorphic to QRDS with $n = 2$ |
| 6 | even | $\{0, 2, 4\}_0$ | $12\alpha + 1$ integer α | 1, 3, 5 | Isomorphic to MQRDS with $n = 2$ |
| 6 | odd | $\{0, 2, 4\}$ | $12\alpha + 3$ integer α | 2, 4 | Isomorphic to residue difference set with $n = 2$ |
| 6 | odd | $\{0, 2, 4\}_0$ | $12\alpha + 3$ integer α | 2, 4 | Isomorphic to modified residue difference set with $n = 2$ |

TABLE 3. List of parameters of QDS and MQDS composed of unions of cyclotomic classes for $n = 4$ and $n = 6$.

6. SUMMARY

This article describes an exhaustive search for QDS and MQDS composed of unions of cyclotomic classes for powers $n = 4, 6, 8$ and 10 . For each value of n studied, some cases were discovered that are simply equivalent to known systems, including QRDS, MQRDS and difference sets. In the case $n = 4$, no new systems were found. However, there were positive new results for $n = 6, 8$ and 10 . In the case $n = 6$ an isolated system for $p = 13$ was found, and for $n = 10$, two MQDS, both for $p = 41$ were discovered, one consisting of a union of two cyclotomic classes and another of four cyclotomic classes. In the two-class case, the choice of $\text{ind}_g 2$ is important. In the case $n = 8$, two entire families of QDS were discovered, the first for all primes of the

| f | C | p | σ | Comments |
|------|-----------------------|------------------------------------|---------------|---|
| even | $\{0, 2\}_0$ | 41 | 5 | single case with g chosen such that $\text{ind}_g 2 \equiv 1$ or $4 \pmod{5}$ |
| even | $\{0, 1, 2, 6\}_0$ | 41 | 5 | single case |
| even | $\{0, 2, 4, 6, 8\}$ | $20\alpha + 1$ integer α | 1, 3, 5, 7, 9 | Isomorphic to QRDS with $n = 2$ |
| even | $\{0, 2, 4, 6, 8\}_0$ | $20\alpha + 1$ integer α | 1, 3, 5, 7, 9 | Isomorphic to MQRDS with $n = 2$ |
| odd | $\{0, 2, 4, 6, 8\}$ | $20\alpha + 3$ integer α | 2, 4, 6, 8 | Isomorphic to residue difference set with $n = 2$ |
| odd | $\{0, 2, 4, 6, 8\}_0$ | $20\alpha + 3$ integer α | 2, 4, 6, 8 | Isomorphic to modified residue difference set with $n = 2$ |

TABLE 4. List of parameters of QDS and MQDS composed of unions of cyclotomic classes for $n = 10$.

form $p = 64z^4 + 128z^3 + 144z^2 + 80z + 17$ and the second for all primes of the form $p = 64z^4 + 48z^2 + 1$ where z is an integer in each case.

APPENDIX - COMPLEMENTARY QDS

Consider a QDS R composed of the union of cyclotomic classes derived from $C = \{c_1, c_2, \dots\}$. Let $A(i)$ be a binary $(0, 1)$ array, defined such that

$$(6.1) \quad \begin{aligned} A(i) &= 1 \text{ if } i \in R \\ A(i) &= 0 \text{ if } i \notin R. \end{aligned}$$

Now define another binary $(0, 1)$ array, $G(j)$ such that

$$(6.2) \quad \begin{aligned} G(j) &= 1 \text{ if } j \in mR \\ G(j) &= 0 \text{ if } j \notin mR. \end{aligned}$$

By the properties of a QDS we have

$$(6.3) \quad \sum_{i=0}^{p-1} A(i)G(i+j) = \begin{cases} N_0 & \text{if } j \equiv 0 \pmod{p} \\ \lambda & \text{if } j \not\equiv 0 \pmod{p} \end{cases}$$

where N_0 is the number of zero differences and λ is the number of non-zero differences between the elements of the sets R and mR . (This two-valued correlation function is useful in applications such as image formation [4, 7, 19] signal processing [17] and aperture synthesis [14]). Now, if we let R^* be the complement of R , we now have $R^* = \{Z_p\} - \{R\}$, composed of the residue classes $C^* = \{Z_n\} - \{C\}$ with the residue zero. Here the corresponding sets $A^*(i)$ and $G^*(j)$ are simply obtained by replacing zeroes for ones in equations (6.1) and (6.2), and vice-versa. It can be readily seen that this transformation has the effect of simply altering the values of the double summation in equation (6.3) to give, say, N_0^* and λ^* . Therefore, we now have a MQDS R^* , which is simply the complement of the original QDS, R .

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