On shifted primes and balanced primes

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Abstract

The asymptotic order of the number of primes, which are such that the shift by a fixed integer is a number supported by a given set of primes times a coprime squarefree number, is determined. The order is also determined when the shift and its negative have this same shape. In each case the order is dependent only on the set of primes and the squarefree core of the shift.

1 Introduction

Given a subset $S \subset \mathbb{N}$ there are a number of standard problems in analytic number theory concerning S. These include finding or bounding the asymptotic density of "shifted primes" $p + k$ which are in S, and the density of double shifted primes $p \pm k$ which are in S, called "balanced primes". In this paper we take S to be first the flat numbers, i.e. numbers which are a power of 2 times an odd squarefree number, and then, more generally, those which can be expressed as a number supported by a given finite set of primes times a squarefree number.

Firstly, some motivation for this study. Dirichlet's classic theorem of 1837 [9], that each arithmetic progression $(-k+nh)_{n\in\mathbb{N}}$ with $(h,k)=1$, has an infinite number of primes and that the proportion of primes is asymptotically an explicit function of h, can be recast in the form that $p+k$ is a multiple of h in this given positive proportion. In the setting of rational primes, the Piatetski-Shapiro prime number theorem of 1953 [23], giving an asymptotic proportion for primes in the sequence $(\lfloor n^c \rfloor)_{n\in\mathbb{N}}$ for $1<c<12/11$, is also a result which we would like to emulate for other subsequences $S\subset\mathbb{N}$.

Subsequent to the work of Dirichlet, there have been a large number of investigations into properties of numbers of the form $p+k$, where k is a non-zero "shift" and p ranges over the set of primes, the so-called "shifted primes". These include Erdos [11], Barban [4], Fogels [14], Heath-Brown [16], Friedlander [15], Hildebrand [17], Elliot [10], Baker and Harman [1], Indlekofer and Timofeev [18], and Banks and Shparlinski [2], to name a sample of works covering over 70 years. For others see the references contained in these works.

In a 1979 paper [12, Theorem 2], Erdös and Odlyzko showed that for any fixed set of primes $P=\{p_1,\ldots,p_l\}$, the set of numbers u, not divisible by any of the p_i, and such that $p-1=p_1^{e_1}\cdots p_l^{e_l}\cdot u$, for some $e_i\geq 0$ where p is prime, has positive lower density. They conjecture that the relative density should be a constant. This question, in the special case $l=1$ and $p_1=2$ was also treated by Chen [7] and Sierpinski [24].

Here we turn the problem around and count the number of primes $p\leq x$ such that $p-1=p_1^{e_1}\cdots p_l^{e_l}\cdot u$, but restrict u, as well as being not divisible by any of the p_i, to being squarefree. Without some restriction on u the problem is without content. (We comment on the question of replacing squarefree by r-free at the end of the paper.) The numbers on the right form a "thick" set of numbers of relative density $1/(\zeta(2)\prod_{p|\wp}(1-1/p^2))$, where $\wp:=p_1\cdots p_l$. The shift -1 is generalized to an arbitrary $k\neq 0$. In this setting we are able to verify the corresponding conjecture to that of Erdös and Odlyzko, namely that the relative density of primes expressible in the given form is a constant, which of course depends on the shift k and the primes P. The constant takes the simple form

$$\prod_{p\nmid k\wp}\left(1-\frac{1}{p^2-p}\right).$$

It is of interest that this relative density depends only on the squarefree core of k. Other types of density identities are intriguing and should be explored.

The method used is based on that developed by Mirsky [20, 21], and requires rather intricate manipulations of modular relationships.

We are also interested in a potential generalization of this result, namely to count primes $p \le x$ such that for given fixed shifts $K = \{k_1, \dots, k_n\}$, for all i, $p + k_i = \alpha_i \cdot u_i$, where the α_i are supported by P and the u_i are squarefree. Although we were not able to make progress with this question (see some remarks at the end of the paper) we did succeed with the special case $K = \{-k, k\}$ of so-called "balanced primes", and find the relative density of primes is given by

$$\prod_{p \nmid k\wp} \left(1 - \frac{2}{p^2 - p}\right).$$

Here, if k is even, we must have $2 \notin P$ and if k is odd $2 \in P$. Otherwise the relative density is zero, and the formula is still correct.

Following many of the theorems there are some sample brief applications. To save space not all of the details of the derivations are always given. However the special case $k \neq 0$, $P = \{2\}$ is worked out in detail. There is a set of specific examples on the author's web site [6]. These are provided to assist the reader who may wish to explore in detail the intricacies of the manipulation of modular relationships involved in the proofs given here.

We use Landau's "O" notation. The implied constant normally depends on all of the parameters, such as the shift and the given set of primes. If P is a possibly empty set of primes, the multiplicative semigroup generated by P, including 1, is denoted $\langle P \rangle$. The symbols p, q, r, with or without subscripts are restricted to taking prime values. By $\{n_1, \dots, n_m\}$ we mean the least common multiple of the positive integers n_i. Finally $\gamma(n) := \prod_{p|n} p$ is the largest squarefree divisor of the natural number n.

The main tool used is the following result for multiple congruences:

Lemma 1 *[21, Lemma A] Let $m_1, \dots, m_n$ be natural numbers and $r_1, \dots, r_n$ corresponding integers. Then a necessary and sufficient condition that the congruences $x \equiv r_j \bmod m_j$ should have a common solution is that*

$$(m_i, m_j) \mid r_i - r_j \text{ for all } 1 \le i < j \le n.$$

In that case in each residue class modulo $\{m_1, \dots, m_n\}$ there is exactly one solution.

2 Flat numbers and primes with flat shifts

We say a natural number n is **flat** if in the standard factorization $n = 2^e p_1 \cdots p_m$ where $e \geq 1$ and $m \geq 0$. See [5]. First note that the number of flat numbers up to x namely $F(x)$, is given for all $x \geq 2$:

$$F(x) = \frac{4}{\pi^2}x + R(x), \ |R(x)| \leq (1+\sqrt{2})\sqrt{x}.$$

To see this let $D(x)$ be the number of odd squarefree numbers. Then $F(x) = D(x/2) + D(x/4) + \cdots$. Also the count of all squarefree numbers is given by $Q(x) = D(x) + D(x/2)$. Let $E(x)$ be the number of even squarefree numbers so $E(2x) = D(x)$. Counting the squarefree numbers in $[x, 2x]$ gives

$$\begin{aligned} (D(2x) - D(x)) + (E(2x) - E(x)) &= Q(2x) - Q(x) \text{ and therefore} \\ D(2x) - D(\frac{x}{2}) &= Q(2x) - Q(x). \end{aligned}$$

Hence

$$D(x) = \sum_{n=0}^{\infty} \left(Q(\frac{x}{4^n}) - Q(\frac{x}{2 \cdot 4^n}) \right).$$

By [22] the error term for $|Q(x) - 6x/\pi^2| \leq 0.5\sqrt{x}$ for $x \geq 8$. Hence $|D(x) - 4x/\pi^2| \leq \sqrt{x}$. Therefore

$$F(x) = \frac{4x}{\pi^2} + R(x) \ \textit{where} \ |R(x)| \leq (1+\sqrt{2})\sqrt{x}.$$

Now let p be an odd prime. We will show there exists a positive integer k such that $p+k$ and $p-k$ are both flat. Let

$$\begin{aligned} A &= \{n : 2 \leq n < p, \ n = p - k \text{ for some odd } k\} \\ B &= \{n : p < n \leq 2p, \ n = p + k \text{ for some odd } k\} \end{aligned}$$

Then $\#A = \#B = (p-1)/2$. By derivation given above, the number of elements of A which are flat is greater than $4p/\pi^2 - (1+\sqrt{2})\sqrt{p} > p/5$ for $p \geq 1051$. For each such element n of A let $\theta(n) = 2p - n$ so $\theta(n) \in B$, and thus there are more than $p/5$ such elements. But the number of elements of B which are not flat is less than $(p-1)/2 - 4p/\pi^2 + (1+\sqrt{2})\sqrt{2p} < p/5$ for $p \geq 1051$. Hence, for that range, there is a k so that $p+k$ and $p-k$ are flat. That the result holds for primes $p < 1051$ is a simple numerical check.

First we count odd primes p which satisfy $p+k$ is flat, where k is a fixed odd integer. That k must be odd follows immediately since $p+k$ must be even.

Theorem 2 *Let $k \geq 1$ be an odd integer, $x > 0$ a real number and $H > 0$ a given constant. Then the number of primes p such that $p+k$ is flat is given by*

$$\#\{p \leq x : p+k \text{ is flat}\} = \prod_{p:(p,2k)=1} \left(1 - \frac{1}{p^2-p}\right) \cdot \mathrm{Li}(x) + O(\frac{x}{\log^H x}).$$

Proof. Let $e \geq 1$ and define the set of primes

$$L_e := \{2 < p \leq x : \exists \text{ odd squarefree } u \text{ with } p-k = 2^e u\},$$

$$\begin{aligned}
\#L_e &= \sum_{\substack{p: p \leq x \\ p-k=2^e u, \\ u \text{ odd and squarefree}}} 1 \\
&= \sum_{p \leq x} \sum_{\substack{a:(a,k)=1, a \text{ odd} \\ p \equiv k \bmod a^2,\ a^2 \leq x/2^e}} \mu(a) + O(1)
\end{aligned}$$

We then split and reverse the sum. (The detailed steps justifying this derivation are similar to those used in the proof [5, Theorem 6], and will not be repeated here or in subsequent proofs.) We subsequently obtain

$$\begin{aligned}
\#L_e &= \left(\sum_{a \geq 1, (a,2k)=1} \frac{\mu(d)}{\phi(2^e a^2)} \right) \cdot \mathrm{Li}(x) + O\left(\frac{x}{\log^{H+1} x}\right) \\
&= \frac{1}{2^{e-1}} \prod_{p:(p,2k)=1} \left(1 - \frac{1}{p^2-p}\right) \cdot \mathrm{Li}(x) + O\left(\frac{x}{\log^{H+1} x}\right).
\end{aligned}$$

Summing over $e \geq 1$ we obtain the stated value of $A(x)$. $\square$

For two simple applications of Theorem 2, note that for all odd k,

$$\prod_{(p,2k)=1} (1 - 1/(p^2 - p)) \geq \prod_{p>2} (1 - 1/(p^2 - p)) > 0.74,$$

so, given any three distinct odd shifts k_1, k_2, k_3, there is a positive proportion of primes p having $p + k_1$, $p + k_2$ and $p + k_3$ all simultaneously flat. With a bit of work it should be possible to increase three to four. Choosing k and $k + 2$ for the shifts we note that $p + k = 2^e u$, $p + k + 2 = 2^f v$ with u, v squarefree, implies $(u, v) = 1$ so $p^2 + 2(k + 1)p + k(k + 1)$ is flat for a positive proportion of primes p. This cannot be extended to all polynomials: for example $(n^2 + 1)^2$ is flat only for $n = 1$ and $n^2 + n + 1$ for no positive integer values of n.

Now we consider odd primes p which satisfy $p \pm k$ are both flat, where $k \geq 1$ is a fixed odd integer.

Theorem 3 *Let $k \geq 1$ be an odd integer, $x > 0$ a real number and $H > 0$ a given constant. Then the number of primes $p \leq x$ such that $p + k$ and $p - k$ are both flat is given by*

$$\#\{p \leq x : p + k \text{ and } p - k \text{ are flat}\} = c_k \mathrm{Li}(x) + O(\frac{x}{\log^H x}).$$

where

$$c_k := \prod_{p:(p,2k)=1} \left(1 - \frac{2}{p^2 - p}\right).$$

Proof. Let $e, f \geq 1$ and define the sets of primes

$$L_e := \{2 < p \leq x : \exists \text{ odd squarefree } v \text{ with } p - k = 2^e v\},$$
$$U_f := \{2 < p \leq x : \exists \text{ odd squarefree } u \text{ with } p + k = 2^f u\}.$$

Then $L_1 \cap U_1 = \emptyset$ and $L_e \cap U_f = \emptyset$ for all $e, f \geq 2$ so we can write

$$B(x) = \{\cup_{f \geq 2} L_1 \cap U_f\} \cup \{\cup_{e \geq 2} U_1 \cap L_e\}$$

where all of the unions are disjoint.

Now fix $e \geq 2$. We will first estimate the size of $U_1 \cap L_e$,where

$$U_1 \cap L_e = \{p \leq x : \exists \text{ odd squarefree } u, v \text{ so } p+k = 2u, p-k = 2^e v\}.$$

$$\begin{aligned}
\# U_1 \cap L_e &= \sum_{\substack{p: p \leq x \\ p+k=2u, \\ p-k=2^e v, \\ u,v \text{ odd and squarefree}}} 1 \\
&= \sum_{p \leq x} \sum_{\substack{(a,b)=1, (ab,2k)=1 \\ p \equiv -k \bmod a^2,\ a^2 \leq x/2 \\ p \equiv k \bmod b^2,\ b^2 \leq x/2^e \\ p \equiv -k+2 \bmod 4 \\ p \equiv k+2^e \bmod 2^{e+1}}} \mu(a)\mu(b) + O(1) \\
&= \sum_{p \leq x} \sum_{\substack{a,b:(a,b)=1, \\ (ab,2k)=1,\ a^2b^2 \leq x^2 2^{-e-1}, \\ p \equiv w \bmod 2^{e+1} a^2 b^2}} \mu(a)\mu(b) + O(1) \\
&= \sum_{p \leq x} \sum_{\substack{(d,2k)=1,\ d^2 \leq x^2 2^{-e-1}, \\ p \equiv w \bmod 2^{e+1} d^2}} \tau^*(d)\mu(d) + O(1)
\end{aligned}$$

where $w = w(d,e)$, the residue obtained through an application of the Chinese Remainder Algorithm, is dependent on d and e. Note also that the LCM $\{a^2, b^2, 4, 2^{e+1}\} = 2^{e+1}a^2b^2$. The function $\tau^*(d)$ is the number of unitary divisors of d, a multiplicative function with $\tau^*(p) = 2$. Note that in the second equation we have used the property that if $g \mid (a,b)$ then $g^2 \mid p$ so $g = 1$ and thus $(a,b) = 1$. If $d = ab$ then necessarily d is odd. Finally we can assume $(d,2k) = 1$ at the cost of $O(1)$ primes.

We then split and reverse the sum and arrive at

$$\begin{aligned}
\# U_1 \cap L_e &= \left(\sum_{d \geq 1, (d,2k)=1} \frac{\tau^*(d)\mu(d)}{\phi(2^{e+1}d^2)} \right) \cdot \mathrm{Li}(x) + O\left(\frac{x}{\log^{H+1} x} \right) \\
&= \frac{1}{2^e} \prod_{p:(p,2k)=1} \left(1 - \frac{2}{p^2 - p} \right) \cdot \mathrm{Li}(x) + O\left(\frac{x}{\log^{H+1} x} \right)
\end{aligned}$$

Summing over $e \geq 2$ and, noticing that the sizes of each corresponding $L_1 \cap U_e$ satisfy the same asymptotic formula as $\# U_1 \cap L_e$, we obtain the stated value of $B(x)$. □

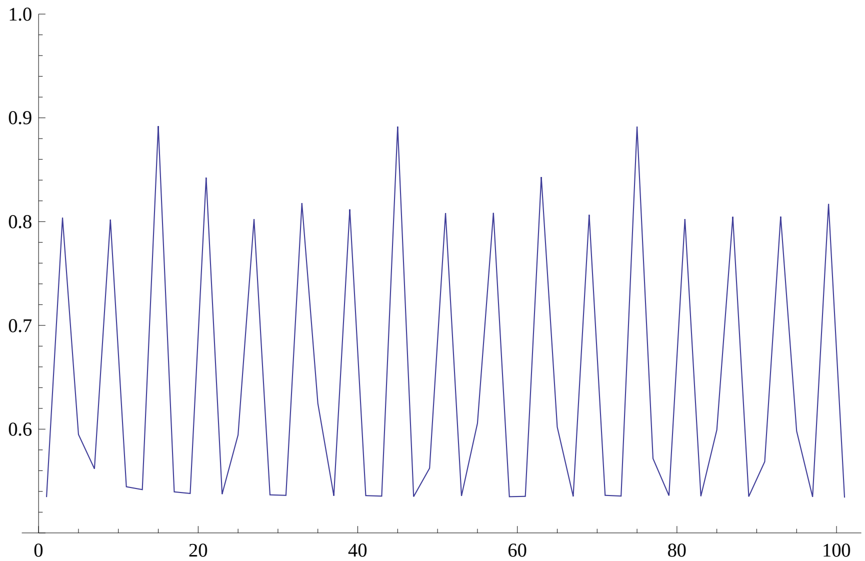

Figure 1: Proportion of primes with $p \pm k$ flat for $1 \leq k \leq 101$

Figure 1 represents a numerical count of primes such that $p \pm k$ are flat, compared with the total number of primes with values $3 \leq p \leq 10^6$, giving close agreement with the result of Theorem 3.

As a simple application of Theorem 3, if $p + k = 2^e u,\ p - k = 2^f v$ and $p > k$ then $(u, v) = 1$, so $p^2 - k^2$ is flat for a definite proportion of more than half of all primes.

3 Single shifts of primes with more general values

A variation of Mirsky's theorem [20] gives the number of primes $p \leq x$ such that $p + k$ is squarefree covering the case where $P = \emptyset$:

Theorem 4 *Let k be a non-zero integer and $H > 0$ a real number. Then*

$$\#\{p \leq x : p + k \text{ is squarefree}\} = \prod_{p:(p,k)=1} \left(1 - \frac{1}{p^2 - p}\right) \cdot \mathrm{Li}(x) + O(\frac{x}{\log^H x}).$$

Now we repeat this theorem, but count only those squarefree integers $u = p + k$ which are coprime with a given fixed integer n.

Theorem 5 *Let k and n be non-zero coprime integers with n squarefree, and $H > 0$ a real number. Then*

$$\#\{p \le x : p+k = u\,,\ u \text{ squarefree, } (u,n) = 1\} = \delta(k,n)\mathrm{Li}(x) + O(\frac{x}{\log^H x})$$

where

$$\delta(k,n) := \prod_{p|n}\left(1 - \frac{1}{p-1}\right) \cdot \prod_{p\nmid nk}\left(1 - \frac{1}{p^2-p}\right).$$

Proof. First let n be odd. Then the number of primes given by the left hand side of the theorem statement is

$$\begin{aligned}
\Sigma_1 &:= \sum_{\substack{p \le x : p+k=u \\ u \text{ squarefree, } (u,nk)=1}} 1 + O(1) \\
&= \sum_{\substack{p : p \le x \\ \forall\ q|n,\ p+k \not\equiv 0 \bmod q}} \ \sum_{a : p \equiv -k \bmod a^2} \mu(a) + O(1) \\
&= \sum_{\substack{a : 1 \le a \\ (a,nk)=1}} \mu(a) \sum_{\substack{p : p \le x \\ p+k \equiv 0 \bmod a^2 \\ p+k \equiv i \bmod q\ \forall\ q|n,\ 1 \le i \le q-1}} 1 + O(1).
\end{aligned}$$

Therefore, performing the same manipulations as performed in previous derivations:

$$\begin{aligned}
\Sigma_1 &= \prod_{q|n}(q-2) \cdot \sum_{1 \le a : (a,nk)=1} \frac{\mu(d)}{\phi(\{a^2, n\})} \cdot \mathrm{Li}(x) + O\left(\frac{x}{\log^H x}\right) \\
&= \prod_{p|n} \frac{p-2}{p-1} \prod_{p \nmid nk}\left(1 - \frac{1}{p^2-p}\right) \cdot \mathrm{Li}(x) + O\left(\frac{x}{\log^H x}\right).
\end{aligned}$$

If n is even then k and u must be odd and there are no odd primes p satisfying $p + k = u$ with $(n, u) = 1$. Therefore the result remains true since $\delta(k, n) = 0$. □

An application of Theorem 5, setting $k = -1$ and letting n be any non-zero odd integer, a positive proportion of primes may be expressed as $p = u + 1$ with u squarefree and $(n, u) = 1$.

Theorem 6 *Let P be a finite set of distinct primes, with $\wp := \prod_{p\in P} p$ and let k be a non-zero integer with $(k,\wp) = 1$. Let $x, H > 0$ be real numbers and let $N(x,k,P)$ be defined as $\#\{p \leq x : p+k = \alpha \cdot u,\ \alpha \in \langle P\rangle, u\ squarefree,\ (u,\wp) = 1\}$. Then we have*

$$N(x,k,P) = \prod_{p\nmid k\wp}\left(1 - \frac{1}{p^2-p}\right) \cdot \mathrm{Li}(x) + O(\frac{x}{\log^H x})$$

where the implied constant depends on k, P and H.

Proof. First we fix $b = p_1^{e_1}\cdots p_l^{e_l}$, $a = q_1\cdots q_m$ where each $e_i \geq 1$, p_i, q_j are distinct primes not dividing k. We count only odd primes p satisfying $p+k = b\cdot u$ where u is squarefree and $(u,kab) = 1$. Let $k = r_1^{s_1}\cdots r_n^{s_n}$. If a is even, k, b and u must be odd so there are at most $O(1)$ primes satisfying $p+k = b\cdot u$. So we assume a is odd. Then

$$\begin{aligned}
\theta(x,k,b,a) &:= \sum_{\substack{p:p\leq x\\ p+k=bu,\ u\ square\ free\\ (u,abk)=1}} 1\\
&= \sum_{\substack{p:p\leq x\\ p+k=bu\\ (u,kab)=1}} \sum_{d:bd^2|p+k} \mu(d)\\
&= \sum_{d\geq 1}\mu(d) \sum_{\substack{p:p\leq x\\ p+k\equiv 0 \bmod bd^2\\ p+k\equiv b,2b,\ldots(p_i-1)b \bmod p_ib,\ 1\leq i\leq l\\ p+k\equiv b,2b,\ldots(q_j-1)b \bmod q_jb,\ 1\leq j\leq m\\ p+k\equiv b,2b,\ldots(r_t-1)b \bmod r_tb,\ 1\leq t\leq n}} 1.
\end{aligned}$$

Thus

$$\begin{aligned}
\theta(x,k,b,a) &= \prod_{p|k}(p-1)\prod_{p|a}(p-2)\prod_{p|b}(p-1)\sum_{\substack{d:d\geq 1\\ (d,kab)=1}} \frac{\mu(d)}{\phi(\{bd^2, b\gamma(k), ba, b\gamma(b)\})}\\
&\times\ \mathrm{Li}(x) + O\left(\frac{x}{\log^{H+1} x}\right)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{\prod_{p|k}(p-1)}{\phi(\gamma(k))} \cdot \frac{\prod_{p|a}(p-2)}{\phi(a)} \frac{\prod_{p|b}(p-1)}{\phi(b\gamma(b))} \prod_{p\nmid kab} \left(1 - \frac{\phi(\gamma(k)ab\gamma(b))}{\phi(p^2\gamma(k)ab\gamma(b))}\right) \\
&\times \ \mathrm{Li}(x) + O\left(\frac{x}{\log^{H+1} x}\right) \\
&= \prod_{p|a}\left(1 - \frac{1}{p-1}\right) \cdot \frac{1}{b} \cdot \prod_{p\nmid kab}\left(1 - \frac{1}{p^2-p}\right) \mathrm{Li}(x) + O\left(\frac{x}{\log^{H+1} x}\right).
\end{aligned}
$$

Now we add the contribution to $N(x,k,P)$ corresponding to each $b \in \langle P \rangle$:

$$
\begin{aligned}
& N(x,k,P) = \\
& \sum_{S\subset P} \prod_{p\in P\setminus S}\left(1 - \frac{1}{p-1}\right) \prod_{\substack{p\nmid k \\ p\notin S \\ p\notin P\setminus S}} \left(1 - \frac{1}{p^2-p}\right) \sum_{e_p\geq 1} \prod_{p\in S} \frac{1}{p^{e_p}} \\
& \times \mathrm{Li}(x) + O\left(\frac{x}{\log^H x}\right) \\
= \ & \sum_{S\subset P} \prod_{p\in P\subset S}\left(1 - \frac{1}{p-1}\right) \prod_{p\in S} \frac{1}{p-1} \prod_{p\nmid k,\ p\notin P}\left(1 - \frac{1}{p^2-p}\right) \\
& \times \mathrm{Li}(x) + O\left(\frac{x}{\log^H x}\right) \\
= \ & \prod_{p\nmid k\wp}\left(1 - \frac{1}{p^2-p}\right) \prod_{p\in S}\left(1 - \frac{1}{p-1} + \frac{1}{p-1}\right) \\
& \times \mathrm{Li}(x) + O\left(\frac{x}{\log^H x}\right) \\
= \ & \prod_{p\nmid k\wp}\left(1 - \frac{1}{p^2-p}\right) \cdot \mathrm{Li}(x) + O\left(\frac{x}{\log^H x}\right).
\end{aligned}
$$

This completes the derivation. □

We give two sample applications of Theorem 6. The proportion of primes of the form $p = 3^e \cdot v + 2$ with $3 \nmid v$ and v squarefree, is asymptotically the same as the proportion of primes of the form $p = 2^e \cdot v + 3$ with $2 \nmid v$

and v squarefree. By taking the set of primes P as an initial sequence of sufficiently many primes we can make the proportion of primes expressible in the form $p = \alpha \cdot u + 1$ arbitrarily close to 1.

4 Double shifted primes with general values

We now proceed to extend Theorem 3 to count odd primes p where $p + k = \alpha \cdot u, p - k = \beta \cdot v$ and where in each case $\alpha, \beta = p_1^{e_1} \cdots p_l^{e_l} \in \langle P \rangle$ for some $e_i' s \geq 0$, u, v are squarefree and coprime to $k\alpha\beta$, and as before, $k, \wp$ are coprime. Many combinations of the $e_i' s$ do not give rise to an infinite set of primes, so initially the situation appears quite complicated.

First we note that if $2 \notin P$ then k must be even and if $2 \in P$ then k must be odd. Otherwise the number of primes satisfying both of the given equalities is finite. To see this, let as before $\wp := \prod_{p \in P} p$ and, suppose that, for some odd prime p,

$$\begin{aligned} p + k &= \alpha \cdot u, \\ p - k &= \beta \cdot v, \end{aligned}$$

where $\alpha, \beta \in \langle P \rangle$ and u, v are squarefree with $(u, \wp) = (v, \wp) = 1$. Note that we can always chose u, v to satisfy this relationship when they exist and then, given p, the values of u, v, α, β are uniquely determined.

Then

$$\begin{aligned} 2k &= \alpha u - \beta v, \\ 2p &= \alpha u + \beta v. \end{aligned}$$

First let $2 \notin P$. Then α, β are odd so u is even iff v is even. In this case there exist odd u', v' so $u = 2u'$, $v = 2v'$ and $p = \alpha u' + \beta v'$, which is impossible. Thus u and v are odd.

If a prime $q \mid (\alpha, \beta)$, then $q = p$ and $p \mid k$ and there are at most $O(1)$ primes in this class, which we ignore. So we can assume $(\alpha, \beta) = 1$. Similarly $(u, v) = 1$. Finally since $p + k = \alpha \cdot u$, which is odd, k must be even, which is what we set out to show.

Now suppose $2 \in P$ and use the same notation as in the previous paragraph. Since $(u, \wp) = (v, \wp) = 1$, u and v are odd. Let $\alpha = 2^e \alpha'$, $\beta = 2^f \beta'$ with $e, f \geq 0$ and α', β' odd. then if a prime $q \mid (\alpha, \beta)$ we have $q \mid 2k$ so $q = 2$ since $(k, \wp) = 1$. Therefore $(\alpha', \beta') = 1$.

If $e \geq 1$ or $f \geq 1$ we can write $2p = 2^e \alpha' u + 2^f \beta' v$ so $e \geq 1$ iff $f \geq 1$, and in this case $p = 2^{e-1} \alpha' u + 2^{f-1} \beta v$ which is odd, so $e = 1$ or $f = 1$ and the other index $f > 1$ or $e > 1$.

If $e = f = 0$ then $p + k = \alpha' u$ implies k is even. However $(k, \wp) = 1$ so this case cannot occur. If $e = 1$ or $f = 1$ then either $p + k$ or $p - k$ is even so necessarily k is odd.

This discussion shows there are two cases to consider: $2 \notin P$ and k even and $2 \in P$ and k odd. In both circumstances we may also assume $(\alpha, \beta) = 1$ and $(u, v) = 1$. We adopt these restrictions in the proof of Theorem 7 below.

Theorem 7 *Let k be an integer and P a non-empty set of primes not dividing k with $2 \in P$ if k is odd and $2 \notin P$ if k is even. Let $\wp$ be the product of the primes in P. Then the number $N(x, k, P)$ of primes $p \leq x$ such that $p + k = \alpha \cdot u$ and $p - k = \beta \cdot v$ where u, v are squarefree with $(u, k\wp) = 1$, $(v, k\wp) = 1$ and $\alpha, \beta \in \langle P \rangle$, is given by:*

$$N(x, k, P) = \prod_{p \nmid k\wp} \left(1 - \frac{2}{p^2 - p}\right) \cdot \mathrm{Li}(x) + O(\frac{x}{\log^H x}).$$

Proof. 1. First assume $P = \emptyset$ and we wish to count primes Σ_1 satisfying $p + k = u$, $p - k = v$ with u, v squarefree. First let k be even. Then

$$\begin{aligned} \Sigma_1 &= \sum_{\substack{p: p \leq x \\ p+k=u \\ p-k=v \\ u, v \text{ squarefree} \\ (u,v)=1}} 1 + \sum_{\substack{p: p \leq x \\ p+k=u \\ p-k=v \\ u, v \text{ squarefree} \\ (u,v) \mid 2k}} 1 \\ &= \sum_{\substack{p: p \leq x \\ p+k=u \\ p-k=v \\ u, v \text{ squarefree} \\ (u,v)=1}} 1 + O(1) \end{aligned}$$

$$= \sum_{\substack{p:p\le x\\ p+k=u\\ p-k=v\\ (u,v)=1}} 1 \sum_{\substack{a^2|u\\ b^2|v\\ (a,b)=1\\ (ab,k)=1}} \mu(a)\mu(b) + O(1)$$

$$= \sum_{p\le x} \sum_{\substack{p+k\equiv 0 \bmod a^2\\ p-k\equiv 0 \bmod b^2\\ (a,b)=1,(ab,k)=1}} \mu(a)\mu(b) + O(1)$$

$$= \sum_{p\le x} \sum_{\substack{p\equiv w \bmod d^2\\ (d,k)=1}} \tau^*(d)\mu(d) + O(1)$$

$$\Sigma_1 = \sum_{d\ge 1,\ (d,k)=1} \frac{\tau^*(d)\mu(d)}{\phi(d^2)} \cdot \mathrm{Li}(x) + O\left(\frac{x}{\log^H x}\right)$$

$$= \prod_{(p,k)=1} \left(1 - \frac{2}{p^2-p}\right) \cdot \mathrm{Li}(x) + O\left(\frac{x}{\log^H x}\right).$$

If k is odd then the product is 0, so the result is still true.

2. Now fix $q > 3$ an odd prime with $q \nmid k$ and count Σ_2, the number of primes $p \le x$ with $p+k = u,\ p-k = v,\ u, v$ squarefree, with the additional constraint $(uv, q) = 1$.

In deriving a formula for Σ_2, the crucial step is seeing that the two sets of congruences $p+k \equiv i \bmod q,\ p-k \equiv j \bmod q$ for $1 \le i, j \le q-1$ gives rise to $q-3$ independent congruences: to see this firstly $k \equiv i \bmod q$ eliminates one from the first and $-k \equiv j \bmod q$ one from the second. Then, using Lemma 1 we must have $q \mid i-k-(j+k)$, so given i, the other index j is determined and vice-versa. Hence

$$\Sigma_2 = \frac{q-3}{\varphi(q)} \prod_{p\nmid kq} \left(1 - \frac{2}{p^2-p}\right) \cdot \mathrm{Li}(x) + O\left(\frac{x}{\log^H x}\right).$$

3. Now fix $m \ge 1$ and consider primes satisfying $p+k = q^m \cdot u,\ p-k = v$, which we denote $\Sigma_3(m)$. Then the corresponding step to that discussed in the previous paragraph, has congruences $p+k \equiv iq^m \bmod q^{m+1},\ p-k \equiv j \bmod q$, $1 \le i, j \le q-1$. So we lose a congruence when $-k \equiv j \bmod q$. But

$(iq^m - k, q^{m+1}) = 1$. so we end up with $q-1$ contributing congruences from this equation. Now, Lemma 1 requires $q \mid q^m i - j - 2k$, so i can be chosen freely, but we must have $j \equiv -2k \bmod q$, fixing j since q is odd. This leads to

$$\Sigma_3(m) \quad = \quad \frac{q-1}{\varphi(q^{m+1})} \prod_{p \nmid kq} \left(1 - \frac{2}{p^2 - p}\right) \cdot \mathrm{Li}(x) + O\left(\frac{x}{\log^{H+1} x}\right).$$

4. Now assume that for all subsets of primes P with $|P| \leq n$, the count Σ_4 of primes up to x satisfying $p + k = \alpha \cdot u,\ p - k = \beta \cdot v$ with $\alpha, \beta \in \langle P \rangle$, and $(k, p) = 1$ for all $p \in P$, satisfies

$$\Sigma_4 = \prod_{p \nmid k\wp} \left(1 - \frac{2}{p^2 - p}\right) \cdot \mathrm{Li}(x) + O\left(\frac{x}{\log^H x}\right).$$

5. For Σ_5, adopt the additional constraint that for some prime $q > 3$ not in P and coprime with k, $p + k = \alpha \cdot u,\ p - k = \beta \cdot v$ and $(uv, q) = 1$. The two additional sets of congruences $p + k \equiv i\alpha \bmod q\alpha,\ p - k \equiv j\beta \bmod q\beta$ for $1 \leq i, j \leq q - 1$ give rise to $q - 3$ independent congruences: to see this firstly $k \equiv i \bmod q$ eliminates one from the first and $-k \equiv j \bmod q$ one from the second. Then, using Lemma 1 we must have $q \mid i - k - (j + k)$ so given i, j is determined and vice-versa.

$$\Sigma_5 = \frac{q-3}{\varphi(q)} \prod_{p \nmid kq\wp} \left(1 - \frac{2}{p^2 - p}\right) \cdot \mathrm{Li}(x) + O\left(\frac{x}{\log^H x}\right).$$

6. Now fix $m \geq 1$ and consider primes satisfying $p + k = q^m \cdot u,\ p - k = v$, which we denote $\Sigma_6(m)$. Then the corresponding step to that discussed in the previous paragraph 5,, has congruences $p + k \equiv iq^m \bmod q^{m+1},\ p - k \equiv j \bmod q$, $1 \leq i, j \leq q - 1$. So we lose a congruence when $-k \equiv j \bmod q$. But $(iq^m - k, q^{m+1}) = 1$ so we end up with $q - 1$ contributing congruences from this equation. Now, Lemma 1 requires $q \mid q^m i - j - 2k$, so i can be chosen

freely, but we must have $j \equiv -2k \bmod q$, fixing j since q is odd. This leads to

$$\begin{aligned} \Sigma_6(m) &= \sum_{n=1}^{\infty} \frac{q-1}{\varphi(q^{n+1})} \prod_{p \nmid kq} \left(1 - \frac{2}{p^2-p}\right) \cdot \mathrm{Li}(x) + O\left(\frac{x}{\log^H x}\right) \\ &= \frac{1}{q-1} \prod_{p \nmid kq} \left(1 - \frac{2}{p^2-p}\right) \cdot \mathrm{Li}(x) + O\left(\frac{x}{\log^H x}\right). \end{aligned}$$

Now assume we have a given set of primes Q with $|Q| = n+1$ and that Q can be written $Q = P \cup \{q\}$ where $q > 3$. (The cases where we must choose $q = 2$ or $q = 3$ are dealt with below.) Then

$$\begin{aligned} N(x,k,Q) &= (\Sigma_1 + 2\Sigma_2) \cdot \mathrm{Li}(x) + O\left(\frac{x}{\log^H x}\right) \\ &= \left(\frac{q-3}{q-1} + 2\frac{1}{q-1}\right) \cdot \prod_{p \nmid kq\wp} \left(1 - \frac{2}{p^2-p}\right) \cdot \mathrm{Li}(x) + O\left(\frac{x}{\log^H x}\right) \\ &= \prod_{p \nmid kq\wp} \left(1 - \frac{2}{p^2-p}\right) \cdot \mathrm{Li}(x) + O\left(\frac{x}{\log^H x}\right) \end{aligned}$$

as claimed. Note the multiplier 2 in the first of these equations covers the possibilities $p+k = \alpha.u,\ p-k = \beta \cdot q^m \cdot v$.

7. In this case $P = \{2\}$ and k is odd so this is the "flat" case and the result follows by Theorem 3.

8. We now consider the case $P = \{3\}$ so k is even. Let $k_1, \ldots, k_s$ be the distinct prime factors of k.

8.1 First we compute $\Sigma_8(l)$ which is the number of odd primes p where there exist u, v squarefree and $p+k = 3^l \cdot u,\ p-k = v$ for fixed $l \geq 1$ and

$(u, 3k) = (v, 3k) = 1$. Then

$$\Sigma_8(l) = \sum_{\substack{p:5\le p\le x\\ p+k=3^l\cdot u\\ p-k=v\\ u,\ v \text{ squarefree}}} 1$$

$$= \sum_{\substack{p:p\le x\\ \forall i,\ p+k\equiv j\cdot 3^l \bmod k_i\cdot 3^l, 1\le j\le k_i-1\\ p+k\equiv 3^l, 2\cdot 3^l \bmod 3^{l+1}\\ \forall i,\ p-k\equiv j \bmod k_i,\ 1\le j\le k_i-1\\ p-k\equiv 1,2 \bmod 3}} \sum_{\substack{a,b:3^l\cdot a^2|p+k\\ b^2|p-k}} \mu(a)\mu(b) + O(1)$$

$$= \sum_{\substack{d:1\le d\\ 2,3\nmid d}} \tau^*(d)\mu(d) \sum_{\substack{p:p\le x\\ p\equiv w \bmod 3^l\cdot d^2\\ \forall i,\ p+k\equiv j\cdot 3^l \bmod k_i\cdot 3^l, 1\le j\le k_i-1\\ p+k\equiv 3^l, 2\cdot 3^l \bmod 3\cdot 3^l\\ \forall i,\ p-k\equiv j \bmod k_i,\ 1\le j\le k_i-1\\ p-k\equiv 1,2 \bmod 3}} 1 + O(1)$$

Thus

$$\Sigma_8(l) = 2\prod_{1\le i\le s}(k_i-1)\cdot \sum_{\substack{d\ge 1\\ 2,3\nmid d}} \frac{\tau^*(d)\mu(d)}{\phi(\{3^l\cdot d^2, k\cdot 3^l, 3\cdot 3^l, k, 3\})}$$
$$\times\ \mathrm{Li}(x) + O\left(\frac{x}{\log^{H+1} x}\right)$$
$$= \frac{2}{\varphi(2\cdot 3^{l+1})}\prod_{p\ne 2,3}\left(1-\frac{2}{p^2-p}\right)\cdot \mathrm{Li}(x) + O\left(\frac{x}{\log^{H+1} x}\right)$$

Note that the second and fourth congruences, using Lemma 1, are not independent and give rise to $\prod_{1\le i\le s}(k_i-1)$ equations, not that quantity squared. Note also that there are two sets of additional congruences, rather than 4, because again applying Lemma 1 to the congruences modulo 3^{l+1} and 3 shows that only one class from the last congruence gives rise to an infinite number of primes. Hence

$$\begin{aligned}\Sigma_8 &= \left(\sum_{l\geq 1}\frac{2}{3^l\cdot 2}\right)\prod_{p\neq 2,3}\left(1-\frac{2}{p^2-p}\right)\cdot \mathrm{Li}(x)+O\left(\frac{x}{\log^H x}\right)\\ &= \frac{1}{2}\prod_{p\neq 2,3}\left(1-\frac{2}{p^2-p}\right)\cdot \mathrm{Li}(x)+O\left(\frac{x}{\log^H x}\right).\end{aligned}$$

8.2 Finally we note that there are at most $O(1)$ odd primes p where there exist u, v squarefree so $p+k=u,\ p-k=v$ and $(uv,3)=1$: either $p+k$ or $p-k$ must be congruent to 0 modulo 3 for $p>3$. Thus

$$N(x,k,\{3\}) = 2\Sigma_8 = \prod_{p\neq 3k}\left(1-\frac{2}{p^2-p}\right)\cdot \mathrm{Li}(x)+O\left(\frac{x}{\log^H x}\right).$$

This completes the derivation in this case, since changing the sign of k indicates the solutions to $p+k=u,\ p-k=3^l\cdot v$ give rise to the same asymptotic expression.

9. Finally we have the case $P=\{2,3\}$ so k is odd. Note only the cases

$$\begin{aligned}&(9.1)\quad p+k=2.u,\ p-k=2^l\cdot 3^m\cdot v,\ l\geq 2,\ m\geq 1\\ &(9.2)\quad p+k=2^l\cdot u,\ p-k=2\cdot 3^m\cdot v,\ l\geq 2,\ m\geq 1,\end{aligned}$$

and their symmetric counterparts, give an infinite number of solutions here, with coefficients denoted C_1 and C_2 respectively.

9.1 First we compute C_1 where first we fix l, m. We skip the details which are similar to other calculations in this paper and focus on the set of congruences which must be satisfied by a prime p if it is to count.

(1) u is odd : $p + k \equiv 2 \bmod 4$
(2) $3 \nmid u : p + k \equiv 2(1, 2) \bmod 2 \cdot 3$
(3) $k_i \nmid u : p + k \equiv 2(1, 2, \ldots, k_i - 1) \bmod 2k_i,\ 1 \le i \le s$
(4) v is odd : $p - k \equiv 2^l \cdot 3^m \bmod 2^{l+1} \cdot 3^m$
(5) $3 \nmid v : p - k \equiv 2^l \cdot 3^m (1, 2) \bmod 2^l \cdot 3^{m+1}$
(6) $k_i \nmid v : p - k \equiv \mathrm{mod} 2^l \cdot 3^m (1, 2, \ldots, k_i - 1) \bmod 2^l \cdot 3^m \cdot k_i,\ 1 \le i \le s.$

Applying Lemma 1 to congruences (3) and (6) enables us to eliminate (6). From (2) and (4) we see that only one of the equations from (2) need be counted. Thus the number of sets of equations is $2 \prod_{1 \le i \le s} (k_i - 1)$.

The argument to Euler's phi function is

$$\{d^2 \cdot 2^{l+1} \cdot 3^m, 2, 6, 2k, 2^{l+1} \cdot 3^m, 2^l \cdot 3^{m+1}, 2^l 3^m k\} = d^2 \cdot 2^{l+1} \cdot 3^{m+1} \cdot k$$

so, summing over the valid values of l and m leads to the coefficient

$$\begin{aligned} C_1 &= \sum_{l=2,m=1}^{\infty} \frac{2}{\varphi(2^{l+1} \cdot 3^{m+1})} \prod_{p \nmid 6k} \left(1 - \frac{2}{p^2 - p}\right) \\ &= \frac{1}{4} \prod_{p \nmid 6k} \left(1 - \frac{2}{p^2 - p}\right). \end{aligned}$$

9.2 Now we compute C_2 using the same simplified approach as for C_1:

(1) u is odd : $p + k \equiv 2^l \bmod 2^{l+1}$
(2) $3 \nmid u : p + k \equiv 1 \cdot 2^l, 2 \cdot 2^l \bmod 3 \cdot 2^l$
(3) $k_i \nmid u : p + k \equiv 1 \cdot 2^l, 2 \cdot 2^l, \cdots, (k_i - 1) \bmod 2^l \cdot k_i,\ 1 \le i \le s$
(4) v is odd : $p - k \equiv 2^1 \cdot 3^m \bmod 2^2 \cdot 3^m$
(5) $3 \nmid v : p - k \equiv 2 \cdot 3^m (1, 2) \bmod 2 \cdot 3^{m+1}$
(6) $k_i \nmid v : p - k \equiv 2 \cdot 3^m (1, 2, \cdots, k_i - 1) \bmod 2 \cdot 3^m \cdot k_i,\ 1 \le i \le s.$

Again using (3) and (6) we can eliminate (6) and from (2) and (4) eliminate one of the equations in (2) leading to the result $C_2 = C_1$.

9.3 Combining the results from 9.1 and 9.2, and including the symmetric patterns, we obtain the overall coefficient in this special case of

$$\prod_{p \nmid 6k} \left(1 - \frac{2}{p^2 - p}\right)$$

and this completes the proof. □

5 Comments

- Generalizing the qualifier squarefree to k-free for $k \geq 2$ in these results should not provide any special difficulties.
- Extending these results to multiple shifts appears to be more challenging. Given a set of non-zero shifts $K = \{k_1, \cdots, k_n\}$ and distinct primes $P = \{p_1, \ldots, p_l\}$, then the initial problem is to determine the relationship between K and P so that there is an infinite number of primes p, (indeed a well determined proportion of all primes given by an infinite product,) satisfying for all $1 \leq i \leq n$

  $$p + k_i = \alpha_i \cdot u_i$$

  with $\alpha_i \in \langle P \rangle$ and u_i squarefree and not divisible by any prime in P. This appears to apply for example to $K = \{3, 7\}$ and $P = \{5, 11\}$, but not to $K = \{3, 5\}$ and $P = \{7, 11\}$ where the relative proportion is apparently 0. It is expected that there should be a well defined asymptotic proportion or zero, but the precise relationship between K and P and the corresponding value of the coefficient has yet to be determined.
- Since we have the result of Heath-Brown [16], that

  $$\#\{p \leq x : p - 1 = 2^e P_2,\ p \equiv u \bmod v\} \gg \frac{x}{\log^2 x}$$

  where $e \in \{1, 2, 3\}$ is fixed, P_2 is prime or the product of two primes and u, v satisfy some restrictions depending on e, we can expect a nice result (i.e., a definite proportion of primes) for the number of primes p up to x with $p+k = 2^e \cdot u$ with e arbitrary and the number of prime factors of squarefree u bounded by some given integer greater than or equal to two. If the values of shifted primes are smooth, and the number of primes tends to infinity with the upper bound on p, we have upper and lower bounds [8, Theorem 1.2], so know the order of magnitude in this case.

- It is also expected that counting primes of the form $p+k = \alpha \cdot u,\ \alpha \in \langle P \rangle$ with u squarefree, in an arithmetic progression $(an+b : n \in \mathbb{N}),\ (a,b) = 1$ should not be too difficult. Compare the work of Banks, Harcharras and Shparlinski [3].

- Extending these ideas to counting values of polynomials $f(n) = \alpha.u$ or $f(p) = \alpha.u$ appears, except in special cases such as the $f(x) = x^2 - k^2$ form treated here, to be much more difficult. Compare Greg Martin's conjectures on the smooth values of polynomials [19] (he obtains exact formulae but uses a variant of the Schinzel-Sierpiński Hypothesis H) and the work of Michael Filaseta, for example [13].

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