AN EXPLICIT BOUND FOR ALIQUOT CYCLES OF REPDIGITS

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Abstract

We find an explicit bound, in terms of g when it is even, for the largest element of an aliquot cycle of repdigits to base g.

1. Introduction

Let $g \geq 2$ be an integer. We say a natural number n is a **repdigit** to base g if there is an integer a with $1 \leq a < g$ and $m \geq 1$ such that $n = a + ag + ag^2 + \cdots ag^{m-1}$. If $a = 1$ then n is called a **repunit**. If $\sigma(n)$ is the sum of divisors function, and we define as usual $s(n) = \sigma(n) - n$, then n is called **perfect** if $s(n) = n$. A finite sequence of distinct integers $\mathcal{C} = \{n_1, \ldots, n_k\}$ is called an **aliquot cycle** if $s(n_i) = n_{i+1}$ for $1 \leq i < k$ and $s(n_k) = n_1$, so a perfect number is just an aliquot cycle of length 1.

Interest in the relationships between repdigits and perfect numbers was initiated by Paul Pollack in [9], who showed that for a given base g there are only a finite number of perfect repdigits to that base, and that the set of all such numbers is effectively computable. Broughan, Guzman Sanchez and Luca [2] found explicit bounds for both the largest perfect repdigit and the number of perfect repdigits to base g. Luca and Te Riele [7] extended the result of Pollack by showing that, at least when the base was even, the number of aliquot cycles of repdigits was finite, and the members of these cycles were all effectively computable. Here we make this result explicit by finding a function of g which gives an upper bound for the cycle with an element of maximum size. The approach taken is to follow the method of [7], making each of the constants explicit. This, on the face of it, requires recourse to results depending on Baker's theory of linear forms in logarithms, so it is expected there would be a lot of scope for reducing the size of the bound, which is exponentially large. This was at least implicit in the work set out in [9, 6, 2].

We use the following notations: $\nu_p(n)$ is the exponent with which the prime p

appears in the factorization of the natural number n, $U_m := (g^m - 1)/(g-1)$ and $V_m := g^m + 1$ for $g \geq 2$ and $m \in \mathbb{N}$. The Landau symbol O depends on g, as do constants c_i, b_i, Δ_i and θ_i for $i = 1, 2, \ldots$. By $\omega(n)$ we mean the number of distinct primes dividing n, by $\tau(n)$ the number of distinct divisors of n, by $\Omega(n)$ the total number of prime divisors of n, including multiplicity, and by $\Omega_g(n)$ the total number of primes, including multiplicity, dividing n which do not divide $g-1$. The expression $p \| n$ means $p \mid n$ and $p^2 \nmid n$ and for $e \geq 1$, $p^e \| n$ means $e = \nu_p(n)$. Following [6] we define $\omega'(n)$ to be the number of distinct odd primes to odd powers in the standard prime factorization of $n \in \mathbb{N}$. The **tower of exponentials** function T is defined as follows: set $T(x) := x$ and

$$T(x_1, \ldots, x_n) := x_1^{T(x_2, \ldots, x_n)}$$

for $n > 1$. For example, $T(x, y, z) = x^{(y^z)}$. Euler's constant γ is defined by

$$\gamma := \lim_{n \to \infty} \left(\sum_{k=1}^{n} \frac{1}{k} - \log n \right).$$

Theorem 1 *Let x be a member of an aliquot cycle of repdigits to base $g \geq 2$ where g is even. Write $x = aU_m$ where $1 \leq a < g$. Then*

$$x \leq T\left(g^2, g^{7s}, g^{7(s-1)}, \ldots, g^{7\cdot 2}, g^7, g\right)$$

where $s := \Omega(m) \leq 2g$.

2. Preliminary Lemmas

In this section, we set out some preliminary lemmas which are needed in the proof of Theorem 1. Of particular importance are Lemma 12 and Lemma 15, which are explicit forms of lemmas of Luca and Pollack.

Lemma 2 *[10, Theorem 3, Corollary] For each $n \in \mathbb{N}$ let p_n be the n'th prime. Then for $n \geq 6$,*

$$p_n < n(\log n + \log\log n).$$

Lemma 3 *[10, Theorem 6, Corollary 1] If $x > 1$ then*

$$\prod_{p \leq x} \frac{p}{p-1} < e^{\gamma} \log x \left(1 + \frac{1}{\log^2 x}\right),$$

where γ is Euler's constant.

Lemma 4 *[10, Theorem 8, Corollary] If $x > 1$ then*

$$\sum_{p \le x} \frac{\log p}{p} < \log x.$$

Lemma 5 *[10, Theorem 5, Corollary] If $x > 1$ then*

$$\sum_{p \le x} \frac{1}{p} < \log\log x + \gamma + \frac{1}{\log^2 x}.$$

Lemma 6 *[4] The only integer solutions to the diophantine equation*

$$\frac{x^n - 1}{x - 1} = \square$$

with $|x| > 1$ and $n > 2$ are

$$\frac{7^4 - 1}{7 - 1} = 2^4 \cdot 5^2 \text{ and } \frac{3^5 - 1}{3 - 1} = 11^2.$$

Next we derive an explicit bound, for a particular Diophantine equation which we need, based on Baker's methods. It is included to give a comparison with the use of the result of Rotkiewicz given below as Lemma 8, which is much better for our purposes.

Lemma 7 *Let $p \ge 5$ be a given prime. Then the Diophantine equation*

$$py^2 = f(x) = 1 + x + x^2 + \cdots + x^{p-1} \tag{1}$$

has at most a finite number of solutions with $x > 1$, and solutions x satisfy

$$x < \exp\exp\exp\left(p^{11p^3}\right)$$

Proof. Let

$$w^2 = h(x) := p + px + px^2 + \cdots + px^{p-1}. \tag{2}$$

Any solution to Equation (1) gives rise to a solution to Equation (2) with $p \mid w$. For (2) the maximum coefficient size $\mathcal{H}$ is p. Since $f(x) = 0$ has degree more than three and all simple roots so does $h(x) = 0$, and we can apply Baker's explicit bound [1] to derive

$$x < \exp\exp\exp\left(\left(p^{10p}\mathcal{H}\right)^{p^2}\right) < \exp\exp\exp\left(p^{11p^3}\right).$$

$\square$

Note for further reference that we need not consider the equation corresponding to (1) for $p = 3$. This is because, taking the two equations $3y^2 = 1 + x + x^2$ and

$x = g^k$, if $k \geq 2$ and g is even, then the left hand side of the first equation is 3 modulo 4, whereas the right is 1. Thus there are no solutions which satisfy these two equations, other than for $k = 1$, and in that case only 1.

The following was proved by Rotkiewicz [11, Theorem 5", Theorem 6'] in greater generality:

Lemma 8 *Let p be an odd prime and x a positive or negative even integer such that $4 \mid x$ or if $2 \| x$ then $p \neq 3$. Then*

$$\frac{x^p - 1}{x - 1} \neq p\square.$$

Lemma 8 enables all of the necessary cases to be covered, unless $p = 3$ and x is twice an odd integer. That case is covered by the following result of Nagell [8].

Lemma 9 *All solutions to the diophantine equation $x^2 + x + 1 = 3y^2$ in positive integers are given by*

$$x_n = \frac{\sqrt{3}}{4}\left(\left(2+\sqrt{3}\right)^{2n+1} - \left(2-\sqrt{3}\right)^{2n+1}\right) - \frac{1}{2}$$

for $n = 0, 1, 2, \ldots$ so $x_0 = 1, x_2 = 22, x_3 = 313$, etc.

Lemma 10 *[6, Lemma 1] Let $\mathcal{P}$ be any finite non-empty set of primes and $\mathcal{P}^*$ the set of positive integers expressible as products of members of $\mathcal{P}$. Then*

$$\sum_{n \in \mathcal{P}^*} \frac{\log n}{n} = \left(\sum_{p \in \mathcal{P}} \frac{\log p}{p-1}\right) \prod_{p \in \mathcal{P}} \left(1 + \frac{1}{p-1}\right).$$

Lemma 11 *[7, Lemma 3] For all $n \in \mathbb{N}$*

$$\nu_2\left(\sigma(n)\right) \geq \sum_{\substack{p \mid n \\ \nu_p(n) \text{ odd}}} \nu_2(p+1).$$

In the following lemma we obtain a big reduction in complexity for the result obtained by Pollack and Luca [6, Lemma 3] wherein the constant -2 replaces their implicit constant $O_g(1)$.

We use the following well known function. If p is a prime not dividing g let the **index of appearance** of p in (U_n), denoted $z(p)$, be the least positive integer d such that $p \mid U_d$. If $p \nmid g - 1$ then $z(p)$ is the multiplicative order of g modulo p and if $p \mid g - 1$ then $z(p) = p$.

Lemma 12 *Let $m \in \mathbb{N}$, $U_m = (g^m - 1)/(g - 1)$ with g even, then*

$$\omega'(U_m) \geq \Omega(m) - 2.$$

Proof. (1) Assume first that m is a power of 2, and write it as $m = 2^s$ with $s \geq 1$. Then

$$U_m = V_{2^0} V_{2^1} V_{2^2} \cdots V_{2^{s-1}}.$$

For $0 \leq i < j < s$, V_{2^i} and V_{2^j} are coprime and odd. For $0 < i < s$, each V_{2^i} is never a square since $g^{2^i} + 1 = x^2 + 1 = \square$ with $x = g^{2^{i-1}}$ has no solution. Hence $\omega'(U_m) \geq s - 1 \geq \Omega(m) - 2$.

(2) Now let $m = 2^s n$ where n is odd and assume that $n > 1$. By repeated application of the identity $U_{2i} = U_i V_i$ we obtain

$$U_m = U_n V_n V_{2n} \cdots V_{2^{s-1}n} = U \cdot V,$$

where $U := U_n$ and $V := U_m/U$. Since g is even U_n is odd, so if a prime $p \mid U_n$, p is odd and $g^n \equiv 1 \pmod{p}$. Thus for $1 \leq i \leq s-1$, $V_{2^i n} \equiv 2 \pmod{p}$ showing that $p \nmid V$. Therefore $(U, V) = 1$.

(3) Now let $n = p_1 \cdots p_k$ be a product of odd primes. We also assume $p_1 \leq p_2 \leq \cdots \leq p_k$. Then, following Pollack and Luca [6, Lemma 3], if we set

$$n = n_1,\ n_2 = \frac{n_1}{p_1}, \ldots n_{i+1} = \frac{n_i}{p_i}, \ldots,$$

with $n_{k+1} = 1$, we can represent U_n as a product of k integers

$$U_n = \frac{U_{n_1}}{U_{n_2}} \frac{U_{n_2}}{U_{n_3}} \cdots \frac{U_{n_k}}{U_{n_{k+1}}} =: T_1 \cdots T_k.$$

(4) We **claim** that for all $i < j$, $(T_i, T_j) = 1$, unless a prime $p \mid g-1$ satisfies $p \mid (T_i, T_j)$ and then we must also have $p_i = p_{i+1} = \cdots = p_j = p$ and $p = (T_i, T_j)$ with $p \| T_l$ for $i \leq l \leq j$.

To derive these properties, let a prime $p \mid (T_i, T_j)$ for some pair (i, j) with $i < j$. Then $n_{i+1} \mid n_i$ so, $p \mid T_j \mid g^{n_j} - 1 \mid g^{n_{i+1}} - 1$ and

$$p \mid T_i = \frac{g^{n_{i+1}p_i} - 1}{g^{n_{i+1}} - 1} = 1 + g^{n_{i+1}} + \cdots g^{n_{i+1}(p_i - 1)}.$$

Reducing this equation modulo p we get $0 \equiv p_i \pmod{p}$ so $p = p_i$. Now since $p \mid T_j \mid U_{n_j}$ the index of appearance of p in (U_n) divides $n_j = p_j \cdots p_k$, i.e. the index is a product of primes greater than or equal to p_j. But since the index is less than or equal to $p = p_i$, it is divisible only by primes less than or equal to p_i. Hence the index must be p_i and $p = p_i = p_{i+1} = \cdots = p_j$ and $p \mid g - 1$. Let l be such that $i \leq l \leq j$ and suppose $x := g^{n_{l+1}}$ so $p \mid x - 1$. Then $\nu_p(T_l) = \nu_p(p) = 1$ so, in particular, $(T_i, T_j) = p$.

(5) Now for each $p \mid (n, g-1)$ let $\mathcal{C}_p := \{i|1 \leq i \leq k,\ p_i = p\}$ and $\mathcal{C}_0 = \{1, \ldots, k\} \setminus \cup_{p|(n,g-1)} \mathcal{C}_p$. If $i \in \mathcal{C}_0$ then T_i is both odd, and by Lemma 6, never a square since g is even. Hence, $\omega'(T_i) \geq 1$. Because for each distinct pair $(i,\ j)$ in $\mathcal{C}_0$ we have $(T_i, T_j) = 1$, we must have $\omega'(\prod_{i \in \mathcal{C}_0} T_i) \geq |\mathcal{C}_0|$.

For the moment fix $p \mid (n, g-1)$ and let $i \in \mathcal{C}_p$. If we now set $x = g^{n_{i+1}}$ and suppose that $4 \mid x$ then $p \mid T_i$ and then, by Lemma 8,

$$T_i = \frac{x^p - 1}{x - 1} \neq p\square.$$

Hence, there is at most one index $i \in \mathcal{C}_p$ with $T_i = p\square$, and that is when $2 \| x$ and $p = 3$, and this can occur on at most one occasion. (Note that the corresponding equation $x^2 + x + 1 = 3y^2$ has an infinite number of solutions by Lemma 9). Therefore $\omega'(\mathcal{C}_p) \geq |\mathcal{C}_p| - 1$ and so

$$\omega'(U_n) \geq |\mathcal{C}_o| - 1 + \sum_{p|(n,g-1)} |\mathcal{C}_p| = \Omega(n) - 1.$$

(6) The next step is along the lines of [6, Proof of Lemma 3] with some amendments. Firstly, in case $g = 2$ and $n = 3$ we have the decomposition

$$V = V_3 V_{2\cdot 3} \cdots V_{2^{s-1}\cdot 3}.$$

The first factor is a square and the remaining factors never square. For $i > 0$, setting $x = 2^{2^i}$, we can write $V_{3\cdot 2^i} = (x+1)(x^2 - x + 1)$, these factors being coprime. The first is not a square by Catalan. The second is also not a square by Lemma 6. Hence in this case we get $\omega'(V) \geq 2(s-1) \geq s-1$ for $s \geq 1$ and $\omega'(V) \geq s - 1$ for $s = 0$. From now in this part we assume $g > 2$.

In the decomposition

$$V = V_n V_{2n} \cdots V_{2^{s-1}n},$$

each $V_{2^i n}$ is odd. If $0 \leq i < s$ and a prime $p \mid V_{2^i n}$, then $g^{n2^i} \equiv -1 \pmod{p}$, so the multiplicative order of g^n modulo p is 2^{i+1}, uniquely determining i. Therefore for $i \neq j$, $V_{2^i n}$ and $V_{2^j n}$ are coprime. If $V_{2^i n} = \square$, then since this is Catalan's equation, we must have $g = 2$, $i = 0$ and $n = 3$. So since $g \geq 4$ we get $\omega'(V_{2^i n}) \geq 1$ for all i with $0 \leq i \leq s$.

Now let q be the smallest prime divisor of n and write

$$V_{2^i n} = \frac{V_{2^i n}}{V_{2^i n/q}} V_{2^i n/q} = \frac{x^q - 1}{x - 1} \cdot V_{2^i n/q},$$

where $x := -g^{2^i n/q}$. Since g is even, by Lemma 6, the first factor is never a square and so $\omega'(V_{2^i n}/V_{2^i n/q}) \geq 1$. If the second factor is a square, by Catalan, since

$g > 2$ we must have $i = 0$, $g = 8$ and $n = q$. So assume $g \neq 8$ or $n \neq q$. We **claim** the two factors on the right are coprime except for at most one index i: if a prime $p \mid (V_{2^i n}/V_{2^i n/q}, V_{2^i n/q})$, then we see that again $g^{2^i n/q} \equiv -1 \pmod p$ so

$$0 \equiv \frac{V_{2^i n}}{V_{2^i n/q}} \equiv 1 + 1 + \cdots + 1 \equiv q \pmod p,$$

giving $p = q$, so as before the index i is uniquely determined, say $i = i_0$. Hence $\omega'(V_{2^i n}) \geq 2$ except for $i = i_0$, and $\omega'(V_{n 2^{i_0}}) \geq 1$. Therefore we have the lower bound

$$\omega'(V) \geq 2(s-1) + 1 = 2s - 1 \geq s - 1.$$

Finally assume $q = n$ and $g = 8$. Consider the decomposition

$$V = (8^q + 1)(8^{2q} + 1)(\cdots)(8^{2^{s-1}q} + 1).$$

As before $\omega'(8^{2^i q} + 1) \geq 2$, for $i > 0$, and we need only show the same inequality holds for $\omega'(8^q + 1)$ for all odd primes q. But $8^3 + 1 = 3^3 \cdot 19$ and for $q > 3$ we can write $8^q + 1 = (2^q + 1)((2^q)^3 - 1)/(2^q - 1))$, which leads to $\omega'(V) \geq 2s \geq s - 1$ for all even $g \geq 2$.

(7) To complete the proof we use the coprime property of the factors of $U_m = UV$ from part (2) and the additivity of $\omega'(m)$ to deduce, using parts (5) and (6)

$$\omega'(U_m) = \omega'(U) + \omega'(V) \geq \Omega(n) - 1 + s - 1 \geq \Omega(m) - 2.$$

This completes the proof. □

Lemma 13 *Let $m \in \mathbb{N}$, $U_m = (g^m - 1)/(g - 1)$ with g even, let a satisfy $1 \leq a \leq g - 1$ and let $x = aU_m$ be a repdigit. Then*

$$\nu_2(\sigma(x)) \geq \Omega(m) - g - 1.$$

Proof. Using Lemma 11 and Lemma 12 we get

$$\begin{aligned} \nu_2(\sigma(x)) = \nu_2(\sigma(aU_m)) &\geq \omega'(aU_m) \geq \omega'(U_m) - \omega'(a) \\ &\geq \omega'(U_m) - g + 1 \geq \Omega(m) - 2 - g + 1 \\ &= \Omega(m) - g - 1. \end{aligned}$$

□

The following is an explicit form for [6, Lemma 2].

Lemma 14 *For all $m \geq 1$ and $g \geq 4$*

$$\log\left(\frac{\sigma(U_m)}{U_m}\right) \leq 1 + 2\log\log\log g + (1+\log\log g)\left(\sum_{d\mid m}\frac{1}{d}\right) + \sum_{d\mid m}\frac{\log d}{d},$$

where the triple logarithm should be replaced by zero when $g \leq 10^3$.

Proof. First write

$$\frac{\sigma(U_m)}{U_m} \leq \prod_{p\mid U_m}\left(1+\frac{1}{p}+\cdots\right) \leq \prod_{p\mid U_m}\left(1+\frac{1}{p-1}\right) \leq \exp\left(\sum_{p\mid U_m}\frac{1}{p-1}\right).$$

Now, since $p \mid U_m$ implies that $z(p) \mid m$, and if $p \nmid g-1$ then $z(p) \mid p-1$, we have

$$\sum_{p\mid U_m}\frac{1}{p-1} \leq \sum_{p\mid g-1}\frac{1}{p-1} + \sum_{\substack{d\mid m\\ d>1}}\left(\sum_{\substack{p\mid U_d\\ p\equiv 1 \pmod d}}\frac{1}{p-1}\right). \tag{3}$$

Fix $d > 1$ such that $d \mid m$. If n is the number of primes p which satisfy $p \mid U_d$ with $p \equiv 1 \pmod d$, then we have $d \leq p$ and so $d^n \leq U_d < g^d$ giving $n \leq d\log g/\log d$. Hence

$$\begin{aligned}\sum_{\substack{p\mid U_d\\ p\equiv 1 \pmod d}}\frac{1}{p-1} &\leq \frac{1}{d}\left(\sum_{1\leq k\leq d\log g/\log d}\frac{1}{k}\right)\\ &\leq \frac{1}{d}\left(1+\log\left(\frac{d\log g}{\log d}\right)\right)\\ &\leq \frac{\log ed}{d}+\frac{\log\log g}{d},\end{aligned}$$

for $g \geq 4$ and (after checking the upper bound explicitly for $d = 2$) for $d \geq 2$.

Therefore, by Equation (3), using Lemma 5, and noticing that in the first term on the right, the number of primes $p \mid g-1$ is not greater than $\log g$, we get, provided $g \geq 16$,

$$\sum_{p\mid g-1}\frac{1}{p-1} \leq 2\sum_{p\leq \log g}\frac{1}{p} < 2\log\log\log g + 2\gamma + \frac{2}{\log^2\log g}.$$

If $g \geq 10^3$ then the sum on the left, evaluating it explicitly, is always bounded by 1. For $g > 10^3$ the sum of the second two terms on the right is also bounded by 1.

Hence we can write

$$\sum_{p|U_m} \frac{1}{p-1} \quad \leq \quad 1 + 2\log\log\log g + \sum_{\substack{d|m \\ d>1}} \left(\frac{\log ed}{d} + \frac{\log\log g}{d} \right).$$

This completes the derivation. □

The second form for the upper bound involves the number of distinct prime divisors of m.

Lemma 15 *Let $m > 1$ and the base $g \geq 4$. Then*

$$\log\left(\frac{\sigma(U_m)}{U_m}\right) \quad \leq \quad 1 + 2\log\log\log g + 4(1+\log\log g)\omega(m) + 4e^{\gamma}\log\left(3\omega(m)\log\left(\omega(m)\right)\right)^2.$$

Proof. By equation (3)

$$\begin{aligned} \sum_{p|U_m} \frac{1}{p-1} \quad &\leq \quad \sum_{p|g-1} \frac{1}{p-1} + \sum_{\substack{d|m \\ d>1}} \left(\frac{\log ed}{d} + \frac{\log\log g}{d} \right) \\ &\leq \quad 1 + 2\log\log\log g + \sum_{\substack{d|m \\ d>1}} \frac{\log d}{d} + 4(1+\log\log g)\omega(m). \end{aligned}$$

where we have used the well known property [3] $\sigma(m)/m < 4\omega(m)$, valid for all $m > 1$.

To bound the middle term we now use [6, Lemma 1], and reprove their Lemma 2 making the constants explicit. Let $\mathcal{P} := \{p_1, \dots, p_k\}$ be the initial sequence of primes with $p_1 = 2$ and let $k = \omega(m)$. Then, by Lemma 2, $p_k \leq 3k\log k$ for $k \geq 2$ and, using Lemma 3, we get

$$\begin{aligned} \prod_{p \leq p_k} \left(1 + \frac{1}{p-1}\right) \quad &\leq \quad e^{\gamma}\log p_k \left(1 + \frac{1}{(\log p_k)^2}\right) \\ &\leq \quad 2e^{\gamma}\log p_k \leq 2e^{\gamma}\log(3k\log k), \end{aligned}$$

where we don't need to consider $k = 1$ since U_m is always odd. Using Lemma 4,

$$\sum_{p \leq p_k} \frac{\log p}{p-1} \leq 2\sum_{p \leq p_k} \frac{\log p}{p} \leq 2\log(3k\log k).$$

Hence, by Lemma 10,

$$\begin{aligned}\sum_{d' \in \mathcal{P}^*} \frac{\log(d')}{d'} &\leq 2e^{\gamma} \log\left(3k \log k\right)\left(2\log(3k\log k)\right) \\ &= 4e^{\gamma} \log\left(3\omega(m)\log\left(\omega(m)\right)\right)^2 .\end{aligned}$$

Finally, combining these bounds we get

$$\begin{aligned}\log\left(\frac{\sigma(U_m)}{U_m}\right) &\leq 1 + 2\log\log\log g + 4(1+\log\log g)\omega(m) \\ &\quad +4e^{\gamma}\log\left(3\omega(m)\log\left(\omega(m)\right)\right)^2 .\end{aligned}$$

□

The final lemma enables us to treat the case of aliquot cycles of just one repdigit.

Lemma 16 *[2, Theorem 1] The largest perfect number x which is a repdigit to base $g \geq 2$ satisfies*

$$x < g^{g^{g^{g^3}}} = T(g,g,g,g^3).$$

3. Proof of Theorem 1

The proof is divided into numbered parts. Assume $\mathcal{C} = \{n_1, \ldots, n_k\}$ is an aliquot cycle with $n_1 < \cdots < n_k$ consisting entirely of repdigits to base g. In part (2) we deal with $g = 2$ and in (3)–(12) we assume $g \geq 4$.

(1) If the cycle has length 1, $\mathcal{C} = \{n_1\}$, then $x := n_1$ is perfect. Therefore, by Lemma 16, we get an explicit upper bound for x in terms of g, namely $T(g,g,g,g^3)$.

(2) In this part we show the case $g = 2$ gives rise to no aliquot cycles. If an aliquot cycle has length 2 or more, let

$$x = U_m,\ y = U_n,\ n \geq m \geq 2.$$

Then $\sigma(x) = x + y$ implies $\sigma(U_m) = 2^m + 2^n - 2 \equiv 2 \pmod 4$. Hence $\sigma(U_m)$ is twice an odd number, which implies $U_m = q^e\square$ with q an odd prime and $e \geq 1$ odd. If $q \equiv 3 \pmod 4$ then we would have $\sigma(q^e) \equiv 0 \pmod 4$, which is not possible. Thus $q \equiv 1 \pmod 4$. The same is true if a cycle has length 1. We now can write $U_m = q\square \equiv 1 \pmod 4$, on the one hand, and $U_m = 2^m - 1 \equiv 3 \pmod 4$ on the other. Therefore there are no aliquot cycles with $g = 2$.

(3) From now on assume $k \geq 2$ and $g \geq 4$. Following [7], let $y := n_k$ and let $x := n_i$ where $i < k$ is such that $s(x) = y$, and m, n are such that $x = aU_m$, $y = bU_n$ with $1 \leq a < g$ and $1 \leq b < g$. Then set

$$c_2 := \left\lfloor \frac{\log(2(g-1))}{\log 2} \right\rfloor + 1 < 2\log g + 1. \tag{4}$$

Under the **assumption** $x > g^{c_2}$ we get $n \geq m \geq c_2$ so, since g is even,

$$(g-1)\sigma(x) = ag^m + bg^n - a - b \equiv -a - b \pmod{2^{c_2}}, \tag{5}$$

and $0 < a + b \leq 2(g-1) < 2^{c_2}$ so therefore $\nu_2(\sigma(x)) < c_2 \leq 2\log g$. If we then use Lemma 13 to write $\nu_2(\sigma(x)) \geq \Omega(m) - g - 1$, we get

$$\omega(m) \leq \Omega(m) \leq g + 2\log g + 1 < 2g =: c_3. \tag{6}$$

Because

$$\frac{\sigma(x)}{x} \leq \frac{\sigma(a)}{a} \cdot \frac{\sigma(U_m)}{U_m} \quad \text{and} \quad \frac{\sigma(a)}{a} \leq \frac{a}{\phi(a)} \leq a \leq g-1,$$

we can write

$$\frac{\sigma(x)}{x} \leq b_1 \frac{\sigma(U_m)}{U_m}, \tag{7}$$

with $b_1 := g - 1 < g$.

(4) For $g \geq 4$ and $m > 1$ we have, by Lemma 15 and Equation (6), the following upper bound:

$$\begin{aligned} \log\left(\frac{\sigma(U_m)}{U_m}\right) &\leq 1 + 2\log\log\log g + 4(1 + \log\log g)\omega(m) \\ &+ 4e^\gamma \log\left(3\omega(m)\log\left(\omega(m)\right)\right)^2 < 92g\log\log g, \end{aligned}$$

and this bound also holds for $m = 1$. Then if we set $c_4 := \exp(94g\log\log g)$ we get $\sigma(x)/x \leq c_4$. Now because

$$c_4 \geq \frac{\sigma(x)}{x} = 1 + \frac{y}{x} = 1 + \left(\frac{b}{a}\right)\left(\frac{g^n - 1}{g^m - 1}\right) \geq 1 + \frac{g^{n-m}}{g-1} \geq g^{n-m-1}, \tag{8}$$

if we set $c_5 := (96g\log\log g)/\log g$ we get $n - m \leq c_5$.

(5) Now we assume that a, b and $c := n - m$ are fixed, noting that $1 \leq a < g$ and $0 \leq n - m \leq c_5$ so a bound for the number of possible values of (a, b, c) is $g^2 c_5$. Let $p(m)$ be the smallest prime dividing m, and **assume** $p(m) > g$. If a prime $q \mid U_m$ then $g^m \equiv 1 \pmod q$ so the multiplicative order of g, $ord_q(g) = e$ say, satisfies $e \mid m$. If $e = 1$then $q \mid g - 1$ so $1 \leq \nu_q(U_m) = \nu_q(m)$, so therefore $q \mid m$ giving $q \geq p(m) > g$. If however $e > 1$ then because $e \mid (m, q-1)$ we have

$q > e \geq p(m) > g$. (The essence of this argument has been given many times.) Therefore, since $a < g$, $(a, U_m) = 1$.

(6) Now

$$\sigma(x) = \sigma(aU_m) = x + y = \left(\frac{a+bg^c}{g-1}\right) g^m - \frac{a+b}{g-1}, \tag{9}$$

so therefore

$$\frac{\sigma(x)}{U_m} = \sigma(a)\frac{\sigma(U_m)}{U_m} = (a+bg^c)\left(1 + \frac{1}{g^m-1}\right) - \frac{a+b}{g^m-1}, \tag{10}$$

and therefore

$$\left|\frac{\sigma(x)}{U_m} - (a+bg^c)\right| \leq b_2\left(\frac{1}{g^m}\right), \tag{11}$$

where we can take $b_2 := 2g^{1+c_5} = 2g\exp(96g\log\log g)$.

(7) Recall $\Omega_g(m)$ is the number of prime factors of m, including multiplicity, which do not divide $g-1$. Since $p(m) > g$, $\Omega(m) = \Omega_g(m)$, and because $f(x) := (\log x)/x$ is a decreasing function of x for $x \geq 3$, we can write

$$\sum_{\substack{d|m \\ d>1}} \frac{\log d}{d} \leq \frac{\tau(m)\log p(m)}{p(m)} \leq 2^{\Omega_g(m)}\frac{\log p(m)}{p(m)} \leq 2^{c_3}\frac{\log p(m)}{p(m)}, \tag{12}$$

using Equation (6). Let $c_6 := 2^{c_3} = 2^{2g}$ and choose $p(m) > c_7$ with $c_7 > g$ so large that

$$c_6\frac{\log p(m)}{p(m)} < \frac{1}{2}.$$

Indeed, we can choose $c_7 = 2^{(7g/2)}$.

Then, because $e^x \leq 1 + 2x$ for $0 \leq x \leq 1/2$, we have

$$\frac{\sigma(U_m)}{U_m} \leq \exp\left(c_6\frac{\log p(m)}{p(m)}\right) \leq 1 + 2c_6\frac{\log p(m)}{p(m)}, \tag{13}$$

and thus, by Equations (11) and (13)

$$|(a+bg^c) - \sigma(a)| \leq 2c_6\sigma(a)\frac{\log p(m)}{p(m)} + \frac{b_2}{g^m},$$

so if we **choose** $m \geq c_5 + 2$ which gives $b_2/g^m \leq \frac{1}{2}$, when it is the case that $\sigma(a) \neq a + bg^c$, we get

$$\frac{\log p(m)}{p(m)} \geq \frac{|(a+bg^c) - \sigma(a)|}{4c_6\sigma(a)} \geq \frac{1}{4c_6 g^2}, \tag{14}$$

whenever $\sigma(a) \neq a+bg^c$. Now for $\epsilon > 0$ and $x > 0$, $(\log x)/x \geq \epsilon$ implies $x < 1/\epsilon^2$. Therefore the inequality of Equation (14) shows that we must have, in the given situation,

$$p_1 := p(m) \leq 16c_6^2 g^4 = 16g^4 2^{4g} \leq 2^{7g} =: \theta_1,$$

so the smallest prime divisor of m is bounded.

(8) Now suppose that $\sigma(a) = a + bg^c$. If also U_m is not prime then the smallest prime divisor of U_m is less than or equal to $\sqrt{U_m} \leq g^{m/2}$. Therefore $\sigma(U_m)/U_m \geq 1 + 1/g^{m/2}$. By Equation (11) we can now write

$$1 + \frac{1}{g^{\frac{m}{2}}} \leq \frac{\sigma(U_m)}{U_m} \leq \frac{a+bg^c}{\sigma(a)} + \frac{b_2}{g^m} = 1 + \frac{b_2}{g^m} \implies m \leq 2\log b_2/\log g, \tag{15}$$

so in this case we have an upper bound for m.

If however U_m is prime then (see [7, Theorem 2]), we claim this case does not arise: if $\sigma(a) = a + bg^c$ with U_m prime, then $\sigma(U_m)/U_m = 1 + 1/U_m$ so, by Equation (10) we get

$$\sigma(a) = \frac{\sigma(a)}{g-1} - \frac{a+b}{g-1} \implies \sigma(a)(g-2) = -(a+b),$$

which is a contradiction, since the left hand side is non-negative and the right strictly negative.

(9) Suppose now that $m = p_1 p_2 \cdots p_s$ with $p_1 \leq p_2 \leq \cdots \leq p_s$, and that for some j with $1 \leq j \leq s-1$ we have established bounds $p_i \leq \theta_i$ for $1 \leq i \leq j$. Fix such a set of primes $\{p_1, \ldots, p_j\}$ and let $m = p_1 \cdots p_j m_j =: n_j m_j$, $g_j := g^{n_j}$, $M_j := (g_j^{m_j} - 1)/(g_j - 1)$, $a_j := a(g_j - 1)/(g-1)$ and note that $n_j \leq \theta_1 \cdots \theta_j$. We will now apply a similar argument, as that which has been used for $j = 1$, to bound $p_1 = p(m)$, to bound $p_{j+1} = p(m_j)$.

(10) We will **assume** $p_{j+1} > g_j = g^{n_j}$ which implies $(a_j, M_j) = 1$, and therefore, using Equation (9), we get

$$\sigma(x) = \sigma(a_j)\sigma(M_j) = \left(\frac{a+bg^c}{g-1}\right) g_j^{m_j} - \frac{a+b}{g-1}. \tag{16}$$

Thus

$$\begin{aligned} \frac{\sigma(x)}{M_j} &= \sigma(a_j)\frac{\sigma(M_j)}{M_j} \\ &= \frac{(a+bg^c)(g_j-1)}{g-1}\left(1 + \frac{1}{g_j^{m_j}-1}\right) - \frac{a+b}{(g-1)M_j} \\ &= \frac{(a+bg^c)(g_j-1)}{g-1} + \Delta_j, \end{aligned}$$

where

$$|\Delta_j| \le \frac{2g^{c_5+n_j}}{g^m} \le b_j \frac{1}{g^m}, \tag{17}$$

with $b_j := 2g^{n_j+c_5} = 2g^{n_j}\exp(96g\log\log g)$.

(11) Now, since $p(m_j) > g_j$, $\Omega(m_j) = \Omega_{g_j}(m_j)$, so as in part (6) we can write

$$\sum_{\substack{d|m_j\\ d>1}} \frac{\log d}{d} \le \frac{\tau(m_j)\log p(m_j)}{p(m_j)} \le 2^{\Omega_{g_j}(m_j)} \frac{\log p(m_j)}{p(m_j)}. \tag{18}$$

As before $c_6 = 2^{2g}$ and choose $p(m_j) > c_7$ with $c_7 > g_j$ so large that

$$c_6 \frac{\log(p(m_j))}{p(m_j)} < \frac{1}{2}.$$

Indeed, we can choose as before $c_7 = 2^{(7g/2)}$.

Then, again as in part (6), we have

$$\frac{\sigma(M_j)}{M_j} \le \exp\left(c_6 \frac{\log p(m_j)}{p(m_j)}\right) \le 1 + 2c_6 \frac{\log p(m_j)}{p(m_j)}. \tag{19}$$

Since

$$\frac{\sigma(a_j)M_j}{M_j} - \sigma(a_j) = \frac{(a+bg^c)(g_j-1)}{g-1} + \Delta_j, \tag{20}$$

if $\sigma(a_j) \ne (a+bg^c)(g_j-1)/(g-1)$ we can write

$$\begin{aligned} 1 \le |(a+bg^c)(g_j-1)/(g-1) - \sigma(a_j)| &\le \sigma(a_j)\left(\frac{\sigma(M_j)}{M_j} - 1\right) + |\Delta_j| \\ &\le 2c_6\sigma(a_j)\frac{\log p(m_j)}{p(m_j)} + |\Delta_j|, \end{aligned}$$

and then if we **choose** m such that $1 + c_5 + n_j \le m$ we get, by Equation (17), $|\Delta_j| \le b_j/g^m \le \frac{1}{2}$, giving

$$\frac{1}{4c_6\sigma(a_j)} \le \frac{\log p(m_j)}{p(m_j)}, \tag{21}$$

whenever $\sigma(a_j) \ne (a+bg^c)(g_j-1)/(g-1)$. Recall that for $\epsilon > 0$, provided $(\log x)/x \ge \epsilon$ we get $x < 1/\epsilon^2$. Therefore the inequality of Equation (21) shows that we must have, in the given situation,

$$p_{j+1} := p(m_j) \le 16c_6^2\sigma(a_j)^2 \le 16 \cdot 2^{4g}a_j^4 \le 16 \cdot 2^{4g}g^4g^{4n_j} \le 2^{7g}g^{4n_j} =: \theta_{j+1}.$$

so the smallest prime divisor of m_j is bounded also.

(12) If $\sigma(a_j) = (a+bg^c)(g_j-1)/(g-1)$, and on the one hand M_j is not prime then the smallest prime factor of M_j is less than or equal to $\sqrt{M_j} \le g_j^{m_j/2}$. Therefore $\sigma(M_j)/M_j \ge 1 + 1/g_j^{m_j/2}$. By Equation (20) we can now write

$$1 + \frac{1}{g_j^{\frac{m_j}{2}}} \le \frac{\sigma(M_j)}{M_j} < 1 + \frac{b_j}{g_j^{m_j}} \implies m_j \le 2\log b_j / \log g_j, \qquad (22)$$

so in this case the proof is complete.

If on the other hand M_j is prime then we will see as before that this case does not arise: if $\sigma(a) = (a+bg^c)(g_j-1)/(g-1)$ with M_j prime, by Equation (16) and a little manipulation we get

$$\sigma(a_j)\frac{g_j - 2}{g_j - 1} = -\frac{a+b}{g-1},$$

which again is a contradiction.

(13) Now we are able to complete the proof. First we derive an upper bound for m, then x, then y. By Equation (6) we have $\Omega(m) \le 2g$. Now $m = p_1 \cdots p_s$ with $s = \Omega(m)$. Looking back through parts (3)-(6), we observe, using the assumption $g \ge 4$,

$$n_1 = p_1 \le \max\left\{2^{7g}, 2 + \frac{96g\log\log g}{\log g}, 2^{\frac{7g}{2}}\right\} \le g^{7g}.$$

Recall that $T(x_1, \ldots, x_n) = x_1^{T(x_2,\ldots,x_n)}$ with $T(x_1) = x_1$. Let

$$B(1) := g^{7g} = T(g^7, g),$$

and assume we have bounds $n_i \le B(i)$ for $1 \le i \le j \le s-1$. Then

$$n_{j+1} \le g^{7jn_j} =: B(j+1).$$

So, if $s = \Omega(m)$, the tower of exponentials $b_m := T(g^{7s}, g^{7(s-1)}, \ldots, g^{7\cdot 2}, g^7, g)$ is a convenient, (but far from best possible) upper bound for m. Then $n+1 \le m + c_5 + 1 \le b_m + (96g\log\log g)/(\log g) + 1$ so $y \le g^{n+1} \le g^{2b_m}$, and this completes the proof.

4. Comment

The three most immediate tasks which arise from this study are (1) reduce if possible the size of the explicit bound, (2) remove the restriction that the base g should be even, and then (3) find an upper bound for a count of all aliquot cycles of repdigits as a function of the base.

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