The Riemann zeta function and holomorphic flows

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Summary

★ The Riemann zeta function $\zeta(s)$ is important,

★ The behavior of $\zeta(s)$ is exceptionally wild,

★ The flow $\dot{z} = f(z)$ can sometimes be used to better understand a meromorphic function $f(z)$ and this applies to $\zeta(s)$ and $\xi(s)$,

★ The symmetrized zeta flow $\dot{s} = \xi(s)$ has periodic orbits and periods associated with each critical simple zero,

★ The logs of these period obey a linear law,

★ The real part of $\zeta(s)$ is better behaved on mysterious "Gram lines".

Zeta's definition

♠ For $\sigma > 1$ let $\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s}$.

♠ For $\sigma > 0$, $\zeta(s) = \sum_{n=1}^{N} \frac{1}{n^s} + \frac{N^{1-s}}{s-1} - s \sum_{n=N+1}^{\infty} \int_0^1 \frac{u}{(u+n)^{s+1}} du$.

♠ For all $s \neq 1$,

$$\zeta(s) = \frac{\Gamma(1-s)}{2\pi i} \int_C \frac{z^{s-1}}{e^{-z} - 1} dz$$

where C is the "keyhole" contour which comes in from the right "below" the x-axis, circles the origin in an anticlockwise direction, then returns to infinity along the "top" of the x-axis.

Zeta's importance

♣ The prime number theorem [Korobov, Vinogradov, 1958]

$$\begin{aligned}\pi(x) &:= \#\{p \leq x : p \text{ is a rational prime}\} \\ &= \mathrm{Li}(x) + \mathrm{O}(x \exp(-A \log^{\frac{3}{5}} x \log\log^{-\frac{1}{5}} x))\end{aligned}$$

♣ If we could show $\zeta(s)$ was non-zero in $[\alpha, 1) \times \mathbb{R}$ for an α with $\frac{1}{2} < \alpha < 1$ then the formula for $\pi(x)$ would have an error $O(x^{\alpha+\epsilon})$.

Wild outside the critical strip

By Kronecker's approximation theorem, if $1 < \sigma$ then

$$\frac{\zeta(2\sigma)}{\zeta(\sigma)} = \inf_t |\zeta(\sigma + it)| < \sup_t |\zeta(\sigma + it)| = \zeta(\sigma)$$

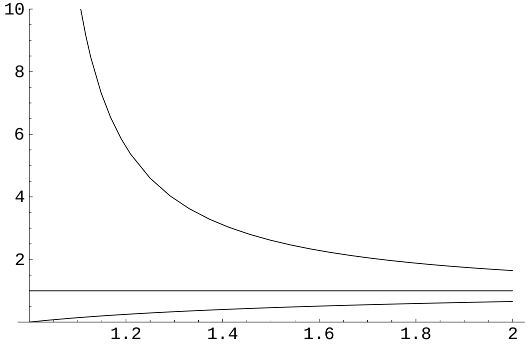

Figure: Plots of $\zeta(2\sigma)/\zeta(\sigma)$ and $\zeta(\sigma)$ for $1 \leq \sigma \leq 2$.

Wild outside and inside the critical strip

• In every strip $[1, 1+\epsilon] \times \mathbb{R}$, $\zeta(s)$ takes on every complex value except 0 an infinite number of times.

• Let $f(s)$ be any continuous non-zero function on $B = B(\frac{3}{4}, \frac{1}{4}) \subset \mathbb{C}$. Then for all $\epsilon > 0$ there is a t so $|\zeta(s+it) - f(s)| < \epsilon$ uniformly on B.

• The curve $\alpha(t) = (\zeta(\sigma + it), \ldots, \zeta^{(n-1)}(\sigma + it))$ is dense in $\mathbb{C}^n$ for any fixed σ with $\frac{1}{2} < \sigma \leq 1$.

Polynomial example for $\dot{z} = f(z)$

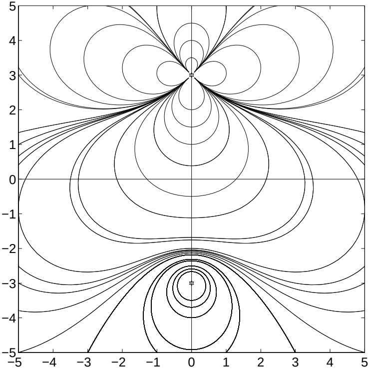

Figure: Holomorphic flow for $\dot{z} = f(z) = (1 + \frac{z}{3i})(1 - \frac{z}{3i})^3$.

$$f'(-3i) = -8i/3,\ f'(3i) = 0,\ f''(3i) = 0,\ f'''(3i) \neq 0.$$

Flow with 5 centers

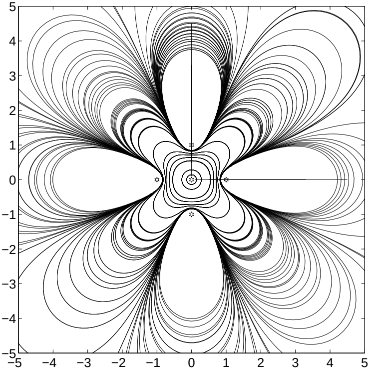

Figure: Flow for $f(z) = iz(z^4 - 1)$.

Classification of zeros

If $f(z_o) = 0$ and $f'(z_o) = \alpha + i\beta \neq 0$ then

- $f'(z_o)$ real implies z_o is a node,
- $f'(z_o)$ pure imaginary implies z_o is a center,
- $f'(z_o)$ has both $\alpha,\ \beta$ non-zero then z_o is a focus,
- at a simple pole the integral curves of $\dot{z} = f(z)$ are those of a saddle.

Limit cycles

• An **orbit** or **trajectory** is a path in $\mathbb{C}$, $t \to \gamma(s_o, t)$ with $\dot{\gamma}(t) = f(\gamma(t))$.

• **Periodic orbits** are trajectories which come back to the initial point s_o in a finite period of time.

• A **limit cycle** is a periodic orbit which has an open neighborhood containing no other periodic orbit.

Theorem

[B, 2003] Let $\Omega \subset \mathbb{C}$ be simply connected and $f : \Omega \to \mathbb{C}$ holomorphic. Then the flow $\dot{z} = f(z)$ has no limit cycle in Ω.

Conjecture: *simply connected* can be replaced by *open*.

Topology of center basins

An orbit γ is a **separatrix** if for some $z \in \gamma$ the maximum interval of existence of the path commencing at z and proceeding in at least one of positive or negative time is finite.

Theorem

Let $\dot{z} = f(z)$ be an entire flow with center at x_o. Let P be the set consisting of x_o together with the union of all of the closed orbits of the flow which contain x_o in their interior. Then P is an open simply connected subset of $\mathbb{C}$ and ∂P consists of the at most countable union of a set of separatrices $\{\gamma(x_\lambda, t) : \lambda \in \Lambda, t \in D_\lambda\}$, D_λ being the maximum interval of existence of the flow through x_λ, where each $\gamma(x_\lambda, t)$ has an unbounded graph.

Example of a center basin

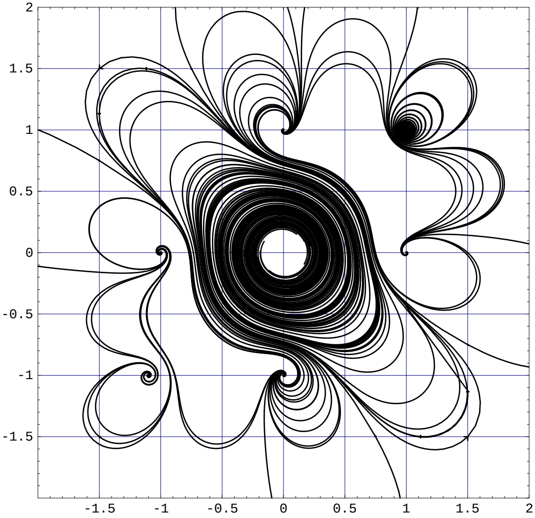

Figure: Flow with two centers.

Structure of a node or focus basin

Theorem

Let $\dot{z} = f(z)$ be an entire flow with a simple zero of at z_o which is a node or a focus. Let P be the set of all points in $\mathbb{C}$ with orbits which tend to z_o in positive time if it is a sink (or in negative time if it is a source). Assume, without loss in generality, that z_o is a sink. Then $P \cup \{z_o\}$ is a simply connected open subset of $\mathbb{C}$ and ∂P consists of an at most countable union of closed connected subsets each being of one of three types: (1) zeros z_1 each with an attached orbit γ_1 such that $L_\alpha(\gamma_1) = z_1$ and $L_\omega(\gamma_1) = \infty$, (2) zeros z_2 each with an attached pair of distinct orbits u, v with $L_\alpha(u) = L_\alpha(v) = z_2$ and $L_\omega(u) = L_\omega(v) = \infty$, and (3) orbits of the form γ_λ where each γ_λ is a positive and negative separatrix.

Example of a focus basin

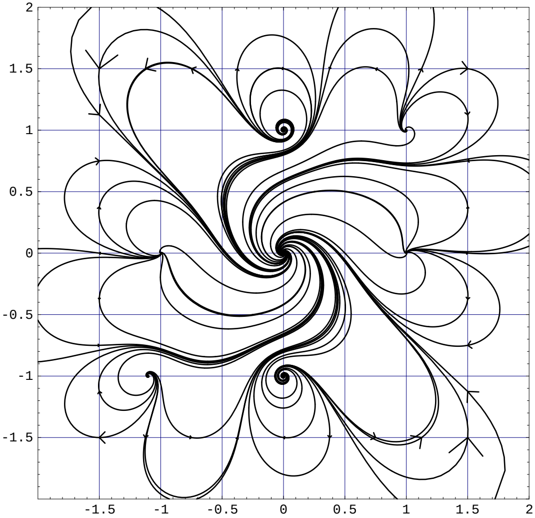

Figure: Neighbourhood of a focus.

Phase portrait for $\zeta(s)$

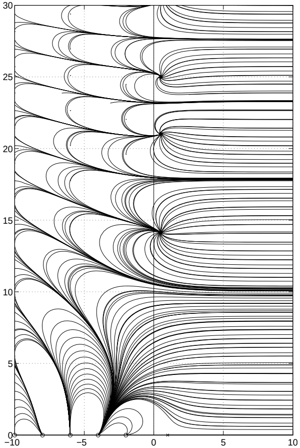

Figure: Lower section of the strip $[-10, 10] \times [0, 30]$.

Near the pole and a real zero

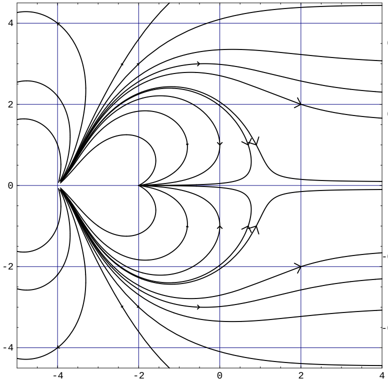

Figure: Region near the pole at $s = 1$.

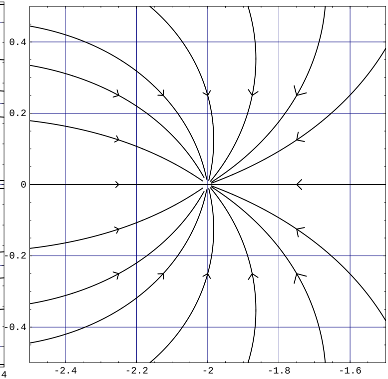

Figure: Region about the sink $s = -2$.

Near two critical zeros

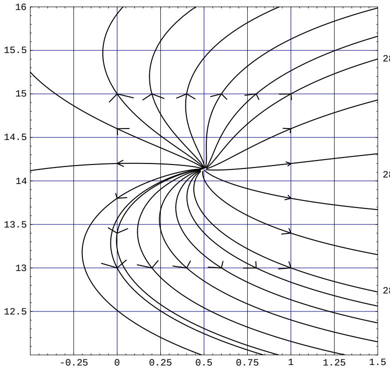

Figure: The first critical zero near $s = 0.5 + 14.1347i$.

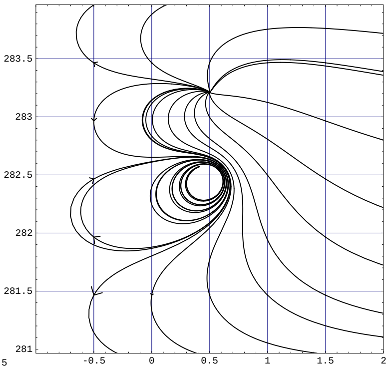

Figure: The first critical sink near $s = 0.5 + 282.465i$.

Phase portrait for $\xi(s)$

$$\xi(s) := \frac{s(s-1)}{2}\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s) \text{ so } \xi(s) = \xi(1-s).$$

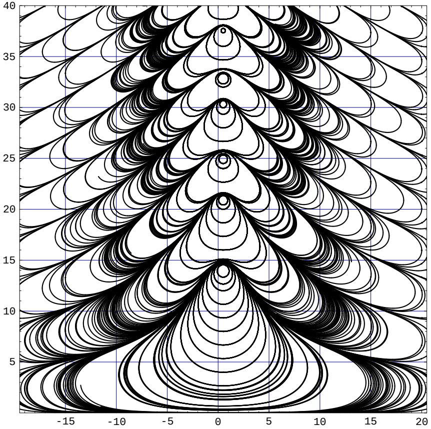

Figure: The phase portrait of $\dot{z} = \xi(z)$ in $[-20, 20] \times [0, 40]$.

A view out to the right for $\xi(s)$

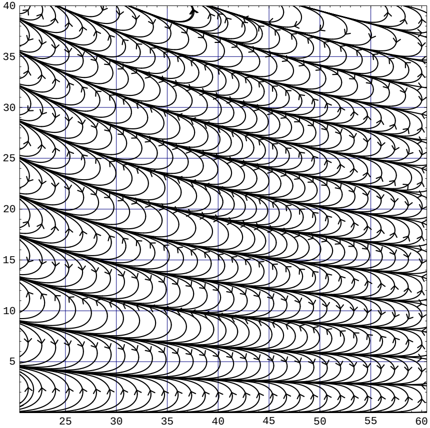

Figure: The phase portrait of $\dot{z} = \xi(z)$ in $[20, 60] \times [0, 40]$.

Properties of the $\dot{s} = \xi(s)$ flow

♣ Each simple critical zero is a center,

♣ There are an infinite number of crossing separatrices,

♣ The separatrices all tend to the x-axis in a manner determined by the gamma factor,

♣ With each simple critical zero there is an associated period, being the transit time on each of the nested periodic orbits.

♣ The flow is "apparently" a deformation of the flow for $\dot{s} = \cosh(s)$

Relationship with the COSH flow

$$\xi(z+\frac{1}{2}) = \eta(z) = \int_1^\infty f(x)\cosh(\frac{1}{2}z\log x)dx$$

$$f(x) = 4\frac{d[x^{3/2}\psi'(x)]}{dx}x^{-1/4},$$

$$\psi(x) = \sum_{n=1}^{\infty} e^{-n^2\pi x}.$$

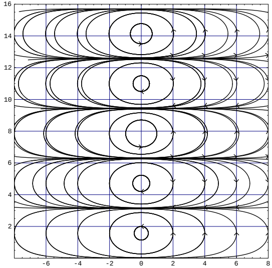

Figure: Phase portrait for $\dot{z} = \cosh(z)$.

Higher up the critical line

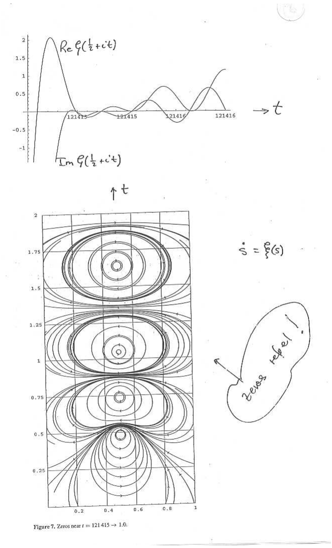

Figure: Zeros near $t = 121415$.

Hypothetical zero configurations

Band number b is the number of separatrices inside a "band" crossing each $x = \sigma > 1$.

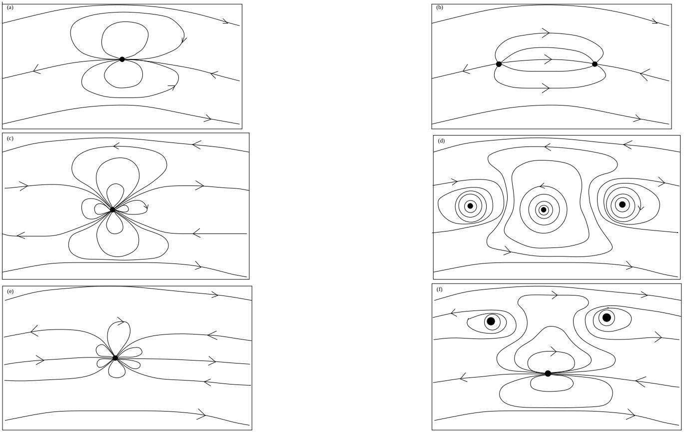

Figure: Hypothetical zero configurations for $b = 1, 2, 3$.

The linear law for the logs of the $\dot{s} = \xi(s)$ periods

Let γ_n be the y-coordinate of a zeta zero. Assume RH and all zeros are simple.

$$
\begin{aligned}
\rho_n &:= \frac{1}{2} + i\gamma_n, \\
T &= \int_\Gamma \frac{ds}{f(s)}, \text{ } \Gamma \text{ is a closed path} \\
T_n &= \pm \frac{2\pi i}{\xi'(\rho_n)}, \\
\log T_n &= \frac{\pi}{4}\gamma_n + O(\log \gamma_n) = RHS,
\end{aligned}
$$

Theorem

[BB,2005] $\log T_n \geq RHS$. Assuming there exists $\theta \geq 0$ such that RH and $|\zeta'(\frac{1}{2} + i\gamma_n)|^{-1} = O(|\gamma_n|^\theta)$ then $\log T_n \leq RHS$.

Plot of the linear law for the log periods

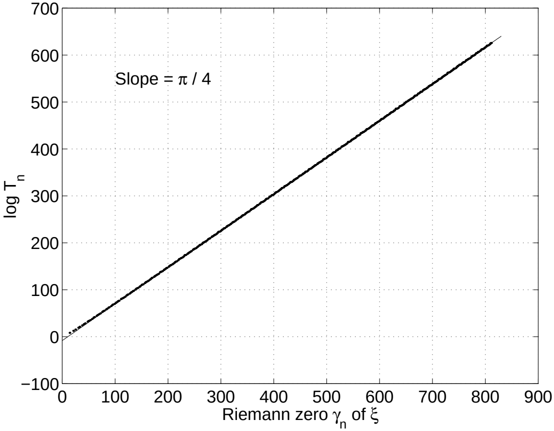

Figure: Plot of logarithm of the period magnitudes against the Riemann zeros up to $\gamma_{502} = 814.1$.

Plot of the residuals

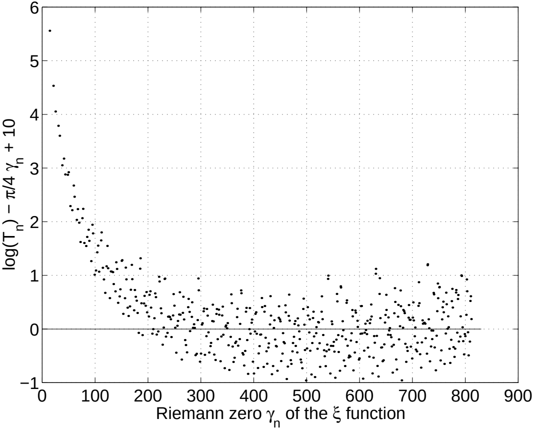

Figure: The residuals using a slope of $\pi/4$.

The Lambda function

Let $\Lambda(s) := 2(2\pi)^{-s}\Gamma(s)\cos(\frac{\pi s}{2})$ so $\zeta(1-s) = \Lambda(s)\zeta(s)$.

♠ The contours $\Im\Lambda(s) = 0$ cut across the critical strip symmetrically.

♠ They cut the critical line at the points $s = \frac{1}{2} + i\gamma$ where $\zeta(s)$ is real (Gram points) or imaginary, with value $\Lambda(s) = \pm 1$. If $\Lambda(s) = 1$ then $\zeta(s)$ is real and if $\zeta(s)$ is a simple zero it is a center. If $\Lambda(s) = -1$ then $\zeta(s)$ is imaginary and if $\zeta(s)$ is a simple zero it is a node.

♠ If $\Im(s) \neq 0$ then $\Lambda(s) \neq 0$.

Flow for $\dot{s} = \Lambda(s)$

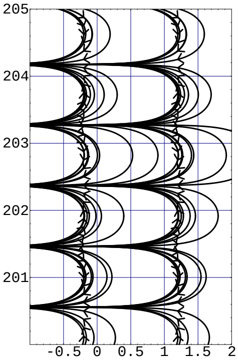

Figure: Phase portrait for $\dot{s} = \Lambda(s)$ with $-1 \leq \sigma \leq 2$ and $200 \leq t \leq 205$.

Equations of the Gram lines

Lemma 1 *The contours of* $\Im\Lambda(s) = 0$*, for* $0 \leq \sigma \leq 1$*, differ from intervals parallel to the x-axis through those points by* $O(1/t)$*. Indeed, the contours satisfy the equations, for* $n \in \mathbb{Z}$*,*

$$\frac{t}{2}\log\frac{t}{2\pi} - \frac{t}{2} - \frac{\pi}{8} + \frac{1}{48t} - \frac{(\sigma - \frac{1}{2})^2}{4t} + O(\frac{1}{t^3}) = \frac{n\pi}{2} \qquad (1),$$

where n is even for lines through Gram points and n is odd for lines through the points where $\zeta(s)$ *is imaginary.*

Extension to the theorem of Titchmarsh

♣ The theorem of Titchmarch [1934]:

$$\lim_{N\to\infty} \frac{1}{N} \sum_{1\le n\le N} [\Re\zeta(\frac{1}{2} + ig_n) - 2] = 0.$$

♣ [BB,2007] If $\frac{1}{2} \le \sigma < 1$, then

$$\lim_{N\to\infty} \frac{1}{N} \sum_{1\le n\le N} [\Re\zeta(\sigma + ig_n) - 1 - (\frac{g_n}{2\pi})^{\frac{1}{2}-\sigma}] = 0.$$

♣ For all σ with $0 < \sigma < 1$ and ϵ with $0 < \epsilon < 1$ the inequality

$$\sum_{1\le n\le N} \Re\zeta(\sigma + ig_n) \ge (1-\epsilon)N$$

holds for all N sufficiently large.

♣ For each σ with $0 < \sigma < 1$ there exist an infinite number of positive integers n with $\Re\zeta(\sigma + ig_n) > 0$.

Motivation: Backlund's method

Let $N(T)$ be the number of zeros of $\zeta(s)$, including multiplicities, in the range $0 < \Im s < T$ and C is the line from $\frac{1}{2} + iT$ to infinity parallel to the x-axis then

$$N(T) = \frac{\vartheta(T)}{\pi} + 1 + \frac{1}{\pi}\Im \int_C \frac{\zeta'(s)}{\zeta(s)} ds$$

If it can be shown that $\Re\zeta$ is positive on C then $\zeta(C)$ never leave the right half plane so the integral term has absolute value less than $\frac{1}{2}$. Hence $N(T)$ is the integer nearest to $\frac{\vartheta(T)}{\pi} + 1$.