Examples for the paper: On shifted primes and balanced primes

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Abstract

These graded examples illustrate the proof techniques in the paper.

1 Single shift examples

In the following set of examples we show how one might extend the range of the shifted decomposition by relaxing the constraint $a \mid A(\mathbf{e})$.

Example 1: Here we count primes p such that $p+1 = 2\cdot 3^2 u$ where u is squarefree and coprime to 2 and 3. We impose the additional constraint that $5 \nmid u$. The leading term of an asymptotic expansion for the number of such primes up to x is $c_1 \cdot Li(x)$ and the aim of the example is to derive $c_1 = 0.039...$

In deriving the leading term we have the summation

$$\Sigma_1 = \sum_{\substack{p: p\le x\\ p+1=18n\\ 5\nmid n,\ 2\nmid n,\ 3\nmid n}} \sum_{d:d^2\mid n} \mu(d)$$

We can assume that p is odd. The condition $5 \nmid (p+1)/18$ is equivalent to $p+1 \equiv 18,\ 55,\ 72 \bmod 90$ where "," represents "or", and where the option

36 is omitted because the only prime solution would be $p = 5$. Similarly $3 \nmid (p+1)/18$ is equivalent to $p + 1 \equiv 18 \bmod 54$. The condition $2 \nmid u$ is represented by restricting d to be odd. Therefore we can write

$$\Sigma_1 = \sum_{\substack{d:d\geq 1 \\ d \ odd}} \mu(d) \sum_{\substack{p\leq x: \\ p+1\equiv 0 \bmod 18d^2 \\ p+1\equiv 18,54,72 \bmod 90 \\ p+1\equiv 18 \bmod 54}} 1.$$

By Lemma **??** there are conditions for the joint congruences in the inner sum to have any solution. For example $(90, 18d^2) \mid 18, 54, 72$ which is the same as $(5, d^2) \mid 1$, so we derive the condition $5 \nmid d$. Each choice of an optional congruence gives rise, asymptotically, to the same number of primes, giving the leading coefficient 3. Hence

$$\begin{aligned}
\frac{\Sigma_1}{Li(x)} &= 3 \cdot \sum_{d\geq 1,\ 2,3,5\nmid d} \frac{\mu(d)}{\phi(\{18d^2, 90, 54\})} \\
&= \frac{1}{24} \cdot \prod_{p\neq 2,3,5} \left(1 - \frac{72}{\phi(\{18p^2, 90, 54\})}\right) \\
&= \frac{1}{24} \cdot \prod_{p>5} \left(1 - \frac{1}{p^2 - p}\right) \\
&= 0.039..
\end{aligned}$$

Example 2: In this example the form is $p + 1 = u$ with u squarefree and not divisible by 3 or 5. We derive the constant $c_2 = 0.177...$

$$\begin{aligned}
\Sigma_1 &= \sum_{\substack{p\le x \\ p+1=u,\ u \text{ squarefree} \\ 3\nmid u,\ 5\nmid u}} 1 \\
&= \sum_{\substack{p:p\le x \\ p+1\equiv 2 \bmod 3 \\ p+1\equiv 2,3,4 \bmod 5}} \sum_{d:d^2|p+1} \mu(d) \\
&= \sum_{\substack{d:1\le d \\ 3\nmid d,\ 5\nmid d}} \mu(d) \sum_{\substack{p:p\le x \\ p+1\equiv 0 \bmod d^2 \\ p+1\equiv 2 \bmod 3 \\ p+1\equiv 2,3,4 \bmod 5}} 1 \\
\frac{\Sigma_1}{Li(x)} &\sim 3\cdot \sum_{\substack{1\le d \\ 3\nmid d,\ 5\nmid d}} \frac{\mu(d)}{\phi(\{d^2,15\})} \\
&= \frac{3}{8}\prod_{p\ne 3,5}\left(1-\frac{1}{p^2-p}\right) \\
&= 0.177..
\end{aligned}$$

Example 3: In this example, as in Example 1, the form is $p+1 = 18u$ with u squarefree and not divisible by 2,3 or 5. We derive the constant $c_3 = 0.039\ldots$ In this case we enforce the requirements $2 \nmid u$ and $3 \nmid u$ by making $6u$ squarefree. Since $5 \nmid u$ we must have $u \equiv 1,2,3,4 \bmod 5$ so $p+1 = 18u \equiv 18,36,54,72 \bmod 90$. But the option $p+1 \equiv 35 \bmod 90$ has no solutions, so is omitted. Again we apply Lemma **??** to show $5 \nmid d$:

$$\begin{aligned}
\Sigma_1 &= \sum_{\substack{p:p\le x\\ p+1=3(6u),\ 6u \text{ squarefree}\\ 5\nmid u}} 1 \\
&= \sum_{\substack{p\le x\\ p+1\equiv 18,54,72 \bmod 90}} \sum_{d:3d^3|p+1} \mu(d) \\
&= \sum_{1\le d} \mu(d) \sum_{\substack{p:p\le x\\ p+1\equiv 0 \bmod 3d^2\\ p+1\equiv 18,54,72 \bmod 90}} 1 \\
\frac{\Sigma_1}{Li(x)} &\sim 3\cdot \sum_{\substack{d\ge 1\\ 5\nmid d}} \frac{\mu(d)}{\phi(\{3d^2,90\})} \\
&= \frac{1}{8}\prod_{p\ne 5}\left(1-\frac{24}{\phi(\{3p^2,90\})}\right) \\
&= 0.039..
\end{aligned}$$

Example 4: In this example the form is $p+7=18u$ with u squarefree and not divisible by 2,3 or 5. We derive the constant $c_4 = c_3 = 0.039...$ We observe that the constant is the same for all prime shifts $k \ge 7$. The only difference between this and Example 3 is in the step where we consider $p+7 \equiv 18, 36, 54, 72 \bmod 90$ the value 72 is discarded rather than 36.

Example 5: In this example the form is $p+5=18u$ with u squarefree and again not divisible by 2,3 or 5. We derive the constant $c_5 = 4c_4/3$. As for Example 4, the only difference between this and Example 3 is the step $p+5 \equiv 18, 36, 54, 72 \bmod 90$ wherein all residues contribute, so the multiplier is 4 rather than 3.

Finally note that if $2 \mid k$ or $3 \mid k$ there are no primes of the form $p+k=18u$ except when $k=15$, wherein the only prime is 3, and $k=16$ wherein the only prime is 2.

2 Double shift examples

We now proceed to count primes p where $p+k = a\cdot u, p-k = b\cdot v$ and where in each case $a, b = p_1^{e_1}\cdots p_l^{e_l}$ for some $e_i's \geq 0$, u is squarefree and coprime to kab, and as before, k, a, b are pairwise coprime. Many combinations of the $e_i's$ do not give rise to an infinite set of primes, so initially the situation seems quite complicated:

Example 7: Let $k = a = 1$ and $B = \{2, 3\}$ so $b = 6$. If $p + k = 2^{e_1}3^{e_2}\cdot u$ and $p - k = 2^{f_1}3^{f_2}\cdot v$, and $p > 3$ then necessarily one of e_2 or f_2 is zero and the other greater than or equal to 1, e_1 or $f_1 = 1$ and the other is strictly greater than 1. To illustrate consider a situation which meets these constraints where we assume the prime p is odd and u, v squarefree:

$$\begin{aligned} p+1 &= 2^3\cdot u, \\ p-1 &= 2\cdot 3^2\cdot v. \end{aligned}$$

$$\begin{aligned}
\Sigma_1 &= \sum_{\substack{p:p\leq x \\ p+1=8u \\ p-1=18v \\ u,\ v \text{ squarefree}}} 1 \\
&= \sum_{\substack{p:p\leq x \\ p+1\equiv 8 \bmod 16 \\ p+1\equiv 8,16 \bmod 24 \\ p-1\equiv 18 \bmod 36 \\ p-1\equiv 18,36 \bmod 54}} \sum_{\substack{a,b:8a^2|p+1 \\ 18b^2|p-1}} \mu(a)\mu(b) \\
&= \sum_{\substack{d:1\leq d \\ 2,3\nmid d}} \tau^*(d)\mu(d) \sum_{\substack{p:p\leq x \\ p\equiv w \bmod 72d^2 \\ p+1\equiv 8 \bmod 16 \\ p+1\equiv 8 \bmod 24 \\ p-1\equiv 18 \bmod 36 \\ p-1\equiv 18,36 \bmod 54}} 1 \\
\frac{\Sigma_1}{Li(x)} &\sim 2\cdot \sum_{\substack{d\geq 1 \\ 2,3\nmid d}} \frac{\tau^*(d)\mu(d)}{\phi(\{72d^2, 16, 24, 36, 54\})} \\
&= \frac{2}{144}\prod_{p\neq 2,3}\left(1 - \frac{2}{p^2-p}\right) \\
&= 0.011..
\end{aligned}$$

Note that in the second step inner sum the two given conditions imply that $(a, b) = 1$ and the residue w is determined by the Chinese Remainder Theorem. In the third step the choice $p + 1 \equiv 16 \bmod 24$ has been omitted, since it gives rise to at most $O(1)$ primes.

Example 8: Keep the same requirements as in the previous example but add the constraint $(u, 5) = 1$, $(v, 5) = 1$. This is reflected in two new equations

$$\begin{aligned} p+1 &\equiv 8, 16, 24, 32 \bmod 40 \\ p-1 &\equiv 18, 36, 54, 72 \bmod 90 \end{aligned}$$

where the residues 16 and 54 can be omitted apriori. Then use Lemma **??** to prune out the sets of congruences which don't have any common solution. This reduces the number of sets of congruences arising from line 3 below from 18 to 4, accounting for the numerator of the leading coefficient.

$$\begin{aligned}
\Sigma_1 &= \sum_{\substack{p:p\le x \\ p+1=8u \\ p-1=18v \\ u,\ v \text{ squarefree}}} 1 \\
&= \sum_{\substack{p:p\le x \\ p+1\equiv 8 \bmod 16,\ p+1\equiv 8,16 \bmod 24 \\ p+1\equiv 8,16,24,32 \bmod 40 \\ p-1\equiv 18 \bmod 36,\ p-1\equiv 18,36 \bmod 54 \\ p-1\equiv 18,36,54,72 \bmod 90}} \quad \sum_{\substack{a,b:8a^2|p+1 \\ 18b^2|p-1}} \mu(a)\mu(b) \\
&= \sum_{\substack{d:1\le d \\ 2,3,5\nmid d}} \tau^*(d)\mu(d) \sum_{\substack{p:p\le x \\ p\equiv w \bmod 72d^2 \\ p+1\equiv 8 \bmod 16 \\ p+1\equiv 8 \bmod 24 \\ p+1\equiv 8,24,32 \bmod 40 \\ p-1\equiv 18 \bmod 36 \\ p-1\equiv 18,36 \bmod 54 \\ p-1\equiv 18,36,72 \bmod 90}} 1 \\
\frac{\Sigma_1}{Li(x)} &\sim 4 \cdot \sum_{\substack{d\ge 1 \\ 2,3,5\nmid d}} \frac{\tau^*(d)\mu(d)}{\phi(\{72d^2, 16, 24, 40, 36, 54, 90\})} \\
&= \frac{4}{576} \prod_{p\ne 2,3,5} \left(1 - \frac{2}{p^2 - p}\right) \\
&= 0.0062..
\end{aligned}$$

Example 9: Let $k=2$ and $B=\{3,5\}$ so $b=15$. If $p+2=2^{e_1}3^{e_2}\cdot u$ and $p-2=2^{f_1}3^{f_2}\cdot v$, and $p>5$ then necessarily at most one of e_1 and f_1 is greater than zero and at most one of e_2 and f_2 is greater than zero. 8.1 First we count odd primes p where there exist u,v squarefree and:

$$\begin{aligned} p+2 &= 3^3\cdot u,\\ p-2 &= 5^2\cdot v. \end{aligned}$$

with $(u,2.3.5)=(v,2.3.5)=1$ so necessarily $(u,v)=1$. Then

$$\begin{aligned}
\Sigma_1 &= \sum_{\substack{p:7\le p\le x\\ p+2=27u\\ p-2=25v\\ u,\ v \text{ squarefree}}} 1\\
&= \sum_{\substack{p:p\le x\\ p+2\equiv 27 \bmod 54\\ p+2\equiv 27,54 \bmod 81\\ p+2\equiv 27,54,81,108 \bmod 135\\ p-2\equiv 25 \bmod 50\\ p-2\equiv 25,50 \bmod 75\\ p-2\equiv 25,50,75,100 \bmod 125}} \sum_{\substack{a,b:27a^2|p+2\\ 25b^2|p-2}} \mu(a)\mu(b)\\
&= \sum_{\substack{d:1\le d\\ 2,3,5\nmid d}} \tau^*(d)\mu(d) \sum_{\substack{p:p\le x\\ p\equiv w \bmod 675d^2\\ p+2\equiv 27 \bmod 54\\ p+2\equiv 27,54 \bmod 81\\ p+2\equiv 54 \bmod 135\\ p-2\equiv 25 \bmod 50\\ p-2\equiv 50 \bmod 75\\ p-2\equiv 25,50,75,100 \bmod 125}} 1\\
\frac{\Sigma_1}{Li(x)} &\sim 8\cdot \sum_{\substack{d\ge 1\\ 2,3,5\nmid d}} \frac{\tau^*(d)\mu(d)}{\phi(\{675d^2,54,81,135,50,75,125\})}\\
&= \frac{8}{5400}\prod_{p\ne 2,3,5}\left(1-\frac{2}{p^2-p}\right)\\
&= 0.0013..
\end{aligned}$$

Note that in the second step inner sum the two given conditions imply that $(a,b)=1$ and the residue w is determined by the Chinese Remainder

Theorem. In the third step the congruences $p+2 \equiv 27 \bmod 135$ and $p-2 \equiv 25 \bmod 75$ have been omitted, since they give rise to at most $O(1)$ primes. The congruences $p-2 \equiv 81, 108 \bmod 135$ have also been omitted because when checked against the last set of congruences they fail condition of Lemma **??**.

Example 10: Now we generalize Example 8. Let $k = 2$ and $P = \{3, 5\}$ so $\wp = 15$.

10.1 First we compute Σ_1 which is the number of odd primes p where there exist u, v squarefree and $p+2 = 3^l \cdot u,\ p-2 = 5^m \cdot v$ for some $l, m \geq 1$ and $(u, 2.3.5) = (v, 2.3.5) = 1$. Then

$$\begin{aligned}
\Sigma_1(l,m) &= \sum_{\substack{p:7\leq p\leq x \\ p+2=3^l\cdot u \\ p-2=5^m\cdot v \\ u,\ v \text{ squarefree}}} 1 \\
&= \sum_{\substack{p:p\leq x \\ p+2\equiv 3^l \bmod 2.3^l \\ p+2\equiv 3^l,2.3^l \bmod 3^{l+1} \\ p+2\equiv 3^l,2.3^l,3.3^l,4.3^l \bmod 5.3^l \\ p-2\equiv 5^m \bmod 2.5^m \\ p-2\equiv 5^m,2.5^m \bmod 3.5^m \\ p-2\equiv 5^m,2.5^m,3.5^m,4.5^m \bmod 5.5^m}} \sum_{\substack{a,b:3^l.a^2|p+2 \\ 5^m.b^2|p-2}} \mu(a)\mu(b) \\
&= \sum_{\substack{d:1\leq d \\ 2,3,5\nmid d}} \tau^*(d)\mu(d) \sum_{\substack{p:p\leq x \\ p\equiv w \bmod 3^l.5^m.d^2 \\ p+2\equiv 3^l \bmod 2.3^l \\ p+2\equiv 3^l,2.3^l \bmod 3.3^l \\ p+2\equiv 2.3^l \bmod 5.3^l \\ p-2\equiv 5^m \bmod 2.5^m \\ p-2\equiv 2.5^m \bmod 3.5^m \\ p-2\equiv 5^m,2.5^m,3.5^m,4.5^m \bmod 5.5^m}} 1 \\
\frac{\Sigma_1(l,m)}{Li(x)} &\sim 8 \cdot \sum_{\substack{d\geq 1 \\ 2,3,5\nmid d}} \frac{\tau^*(d)\mu(d)}{\phi(\{3^l.5^m.d^2, 2.3^l, 3.3^l, 5.3^l, 2.5^m, 3.5^m, 5.5^m\})} \\
&= \frac{8}{\varphi(2.3^{l+1}.5^{m+1})} \prod_{p\neq 2,3,5} \left(1 - \frac{2}{p^2-p}\right)
\end{aligned}$$

Hence

$$\begin{aligned}
\frac{\Sigma_1}{Li(x)} &\sim \left(\sum_{l,m\geq 1}\frac{8}{3^l.5^m(2.4)}\right)\prod_{p\neq 2,3,5}\left(1-\frac{2}{p^2-p}\right)\\
&= \frac{1}{8}\prod_{p\neq 2,3,5}\left(1-\frac{2}{p^2-p}\right)\\
&= 0.112..
\end{aligned}$$

10.2 Now count odd primes p where there exist u, v squarefree and $p+2 = 3^l 5^m \cdot u,\ p-2 = v$ for some $l, m \geq 1$. We spare the reader the details, but in step 3, for fixed l, m, we arrive at 8 sets of congruences, the same number as in 9.1 (but different congruences). Hence $\Sigma_2 = \Sigma_1$

10.3 Next we count odd primes p where there exist u, v squarefree and some $l \geq 1$ so $p+2 = 3^l \cdot u,\ p-2 = v$. Here the pruning of the congruences at step three using Lemma **??** depends on the equivalence class of l modulo 4, but, fortunately, we are left with the same number, 4 of valid classes in each case. The reader is spared the details. This leads to

$$\begin{aligned}
\frac{\Sigma_3(l)}{Li(x)} &\sim 4\cdot\sum_{\substack{d\geq 1\\ 2,3,5\nmid d}}\frac{\tau^*(d)\mu(d)}{\phi(\{3^l.d^2, 2.3^l, 3.3^l, 5.3^l, 2, 3, 5\})}\\
&= \frac{4}{\varphi(2.3^{l+1}.5)}\prod_{p\neq 2,3,5}\left(1-\frac{2}{p^2-p}\right)
\end{aligned}$$

Hence

$$\begin{aligned}
\frac{\Sigma_3}{Li(x)} &\sim \left(\sum_{l\geq 1}\frac{4}{8.3^l}\right)\prod_{p\neq 2,3,5}\left(1-\frac{2}{p^2-p}\right)\\
&= \frac{1}{4}\prod_{p\neq 2,3,5}\left(1-\frac{2}{p^2-p}\right)\\
&= 0.224..
\end{aligned}$$

10.4 Note that there are *no* odd primes $p > 5$ such that there exist u, v squarefree so $(u, 30) = 1$, $(v, 30) = 1$ and for some $l \geq 1$: $p + 2 = 5^l \cdot u$, $p - 2 = v$, since the second equation gives $p \equiv 1 \bmod 3$ so by the first $3 \mid u$. Now note also that there are *no* primes $p > 5$ where there exist u, v squarefree so $(u, 30) = 1$, $(v, 30) = 1$ and $p + 2 = u$, $p - 2 = v$.

10.5 Finally we combine the computations in 10.1-10.5 to count the primes $p > 5$ up to x satisfying

$$\begin{aligned} p + 2 &= \alpha \cdot u, \\ p - 2 &= \beta \cdot v \end{aligned}$$

where $P = \{3, 5\}$ and $\alpha, \beta \in \langle P \rangle$ and $(uv, 30) = 1$:

$$\begin{aligned} \frac{\Sigma}{Li(x)} &= \frac{2(2\Sigma_1 + \Sigma_3)}{Li(x)} \\ &= \prod_{p \neq 2,3,5} \left(1 - \frac{2}{p^2 - p}\right) \\ &= 0.89.. \end{aligned}$$

where the leading coefficient has simplified to 1! Numerical evaluation of other examples indicates that this formula should hold more generally.