Small prime gaps

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Abstract

One of the more spectacular recent advances in analytic number theory has been the proof of the existence of an infinite number of pairs of prime numbers at a constant small distance apart. Work on this goes back many years. An illustrated overview of developments will be given, tracing through the work of Erdos, Bombieri/Davenport, Goldson/Pintz/Yildrim, Zhang, Tao/Polymath8a/b, and Maynard. A recent breakthrough by students of Ken Ono will also be described.

$$n! + 2, \ldots, n! + n$$

are $n - 1$ composites $\implies \limsup_{m\to\infty} p_{m+1} - p_m = \infty$

We expect $\liminf_{m\to\infty} p_{m+1} - p_m = 2$, or there are an infinite number of integers m such that $p_{m+1} - p_m = 2$, the twin primes conjecture.

September 2016 the largest known pair of twins had 388342 digits:

$$2996863034895 \times 2^{1290000} \pm 1.$$

Brun showed in the early 1900's that if (p_{n_j}, p_{n_j+1}) are all the twins starting with $(3, 5)$

$$\sum_{j \in \mathbb{N}} \left(\frac{1}{p_{n_j}} + \frac{1}{p_{n_j+1}} \right) = \left(\frac{1}{3} + \frac{1}{5} \right) + \cdots < \infty.$$

and that the number of twins up to x, $\pi_2(x)$ had an upper bound as $x \to \infty$

$$\pi_2(x) \ll \frac{x}{\log^2 x}$$

for some absolute constant, later conjectured by Hardy and Littlewood to be $2C_2$ where C_2 is the famous twin primes constant:

$$C_2 := \prod_{p \geq 3} \left(1 - \frac{1}{(p-1)^2} \right) = 0.66016181...$$

Average Gap

There are approximately $x/\log x$ primes up to x. Thus the average gap is $\log x$. Can we sometimes get smaller gaps than this average?

year	bound	people
1926	$(\frac{2}{3} + o(1)) \log x$	Hardy/Littlewood/GRH
1940	$(\frac{3}{5} + o(1)) \log x$	Rankin/GRH
1940	$(1 - c + o(1)) \log x$	Erdős
1954	$(\frac{15}{16} + o(1)) \log x$	Ricci
1966	$(0.4665 + o(1)) \log x$	Bombieri/Davenport
1972	$(0.4571 + o(1)) \log x$	Pil'tjai
1975	$(0.4542 + o(1)) \log x$	Uchiyama
1973	$(0.4463 + o(1)) \log x$	Huxley
1977	$(0.4425 + o(1)) \log x$	Huxley
1984	$(0.4394 + o(1)) \log x$	Huxley
1988	$(0.2484 + o(1)) \log x$	Maier

Table: Time line of decreasing prime gaps I.

Enter Goldston/Pintz/Yildrim

year	bound	people
2006	$o(\log x)$	Goldston/Pintz/Yildirim
2009	$C(\log x)^{\frac{1}{2}}(\log\log x)^2$	Goldston/Pintz/Yildirim
2013	7.0×10^7	Zhang
2013	4680	Polymath8a
2013	600	Maynard
2014	246	Polymath8b

Table: Time line of decreasing prime gaps II.

Who are Goldston/Pintz/Yildrim (GPY)?

Figure: Dan Goldston (1952–), Janos Pintz (1950–) and Cem Yildirim (1961–).

What did GPY do I?

They commence with an admissible k-tuple $\mathscr{H} = \{h_1, \ldots, h_k\}$. The goal is to show that there are an infinite number of integers n such that $n + \mathscr{H}$ contains at least two primes. This will be so if for all $N \in \mathbb{N}$ sufficiently large, we can find at least two primes in $n + \mathscr{H}$ for some n with $N < n \le 2N$.

Let $\Lambda(p^m) := \log p$ and $\Lambda(n) = 0$ if n is not a prime power.

Their key idea, is that if for some $n \in (N, 2N]$ with $n > h_k$ and $\rho \in \mathbb{N}$ we had

$$\left(\sum_{j=1}^{k} \Lambda(n + h_i) \right) - \rho \log(3N) > 0$$

then since each $\Lambda(n + h_j) < \log(3N)$ we must have at least $\rho + 1$ of the $\Lambda(n + h_j) > 0$. so at least $\rho + 1$ primes in $n + \mathscr{H}$.

What did GPY do II?

Introducing a non-negative weight function $w(n)$

$$S_1(N) := \sum_{n=N+1}^{2N} \left(\sum_{j=1}^{k} \Lambda(n + h_j) - \rho \log(3N) \right) w(n)$$

so, if for all N sufficiently large, $S(N) > 0$ we must have, for at least one n with $N < n \leq 2N$, both $w(n) > 0$ more than ρ primes in $n + \mathscr{H}$.

Finding a viable definition for the weights $w(n)$ is difficult. Using squares makes the non-negativity criteria easy to attain, but evaluating the the two sums $\sum_{n \leq N} \Lambda(n + h_j) w(n)$ and $\sum_{n \leq N} w(n)$ goes to the heart of the challenge this problem provides.

What did GPY do III?

GPY "went global" making the weights depend on all k-tuples with $h_k \leq h$, h being a fixed but arbitrary parameter. They also "went local" by truncating the von Mangoldt function, reducing the number of terms in the divisor sum representation, and introducing a truncation level $R \ll N^{\frac{1}{4k}-\epsilon}$. They use

$$\Lambda(n) = \sum_{d|n} \mu(d) \log\left(\frac{n}{d}\right) \to \Lambda_R(n) := \sum_{\substack{d|n \\ d \leq R}} \mu(d) \log\left(\frac{R}{d}\right),$$

and

$$\Lambda_R(n, \mathscr{H}) := \Lambda_R(n+h_1) \cdots \Lambda_R(n+h_k),$$

and then define

$$w(n) := \sum_{\substack{(h_1,\ldots,h_k) \in [1,h]^k \\ h_j \text{ distinct}}} \Lambda_R(n, \mathscr{H})^2.$$

Then $S_2(N) := \sum_{n=N+1}^{2N} \left(\sum_{1 \leq h_0 \leq h} \Lambda(n+h_0) - \rho \log(3N) \right) w(n).$

$$\text{Let } \Delta := \liminf_{n\to\infty} \frac{p_{n+1} - p_n}{\log p_n}$$

Recall $h_k \leq h$. To get $S_2(N) > 0$ with $\rho = 1$ requires $h \geq \frac{3}{4} \log N$. Thus

$$\Delta = \liminf_{n\to\infty} \frac{p_{n+1} - p_n}{\log p_n} \leq \liminf_{N\to\infty} \frac{h}{\log N} = \frac{3}{4} \implies \Delta \leq \frac{3}{4},$$

A distinct disadvantage of this approach is that to form the weights many divisor sums are multiplied together, forcing them to be made very short, i.e. choosing R very small relative to N.

What did GPY do V?

GPY overcame the many short divisors problem by defining a polynomial

$$P_{\mathscr{H}}(n) = (n+h_1)\cdots(n+h_k),$$

and then using the generalized von Mangoldt function $\Lambda_k(n)$, which is zero if n has more than k prime factors.

$$\text{Let } \Lambda_k(n) := \sum_{d|n} \mu(d)\left(\log \frac{n}{d}\right)^k,$$

Then $\Lambda_k(P_{\mathscr{H}}(n)) \neq 0$ means each of the translates $n + h_j$ must be a prime (power).

$$\text{Truncating } \Lambda_R(n, \mathscr{H}) := \frac{1}{k!} \sum_{\substack{d|P_{\mathscr{H}}(n) \\ d \leq R}} \mu(d)\left(\log \frac{R}{d}\right)^k.$$

This allows us to detect a prime tuple with a single divisor sum. However when applied it resulted in $\Delta \leq 0.1339$.

What did GPY do VI?

The new idea was to introduce an additional parameter l, as small as possible, with $1 \leq l \leq k-2$ and consider k-tuples $\mathcal{H}$, detecting prime (powers) with

$$\Lambda_R(n, \mathcal{H}, l) := \frac{1}{(k+l)!} \sum_{\substack{d \mid P_{\mathcal{H}}(n) \\ d \leq R}} \mu(d) \left(\log \frac{R}{d} \right)^{k+l}.$$

Then if $\Lambda_R(n, \mathcal{H}, l) \neq 0$, $\mathcal{H}$ must have at least $k-l \geq 2$ prime terms, so we are done.

The expression to check for positivity is then

$$S_3(N) := \sum_{n=N+1}^{2N} \left(\sum_{1 \leq h_0 \leq h} \Lambda(n+h_0) - \rho \log(3N) \right) \sum_{\substack{(h_1, \ldots, h_k) \in [1,h]^k \\ h_j \text{ distinct}}} \Lambda_R(n, \mathcal{H}, l)^2,$$

and this enabled $\Delta = 0$ to be deduced, using a much greater value of $R \ll N^{\frac{1}{4}-\epsilon}$.

Zhang and his wonderful breakthrough

An unlikely creative genius

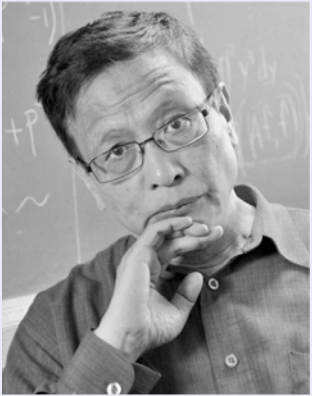

Figure: Yitang Zhang (1955 –)

Zhang's theorem

There are an infinite number of primes such that $p_{n+1} - p_n \leq 7 \times 10^7$.


Enrico Bombieri (1940–)

Let $A > 0$ and if $B := 2A + 5$ let $Q := \sqrt{x}/(\log x)^B$. Then there is a constant $C > 0$ such that

$$\sum_{q \le Q} \max_{\substack{a \bmod q \\ (a,q)=1}} \left| \theta(x, q, a) - \frac{x}{\varphi(q)} \right| \le \frac{Cx}{\log^A x}.$$

$$\implies \frac{\sum_{q \le Q} \max_{\substack{a \bmod q \\ (a,q)=1}} \left| \theta(x, q, a) - \frac{x}{\varphi(q)} \right|}{Q} \le \frac{C\sqrt{x}}{\log^{B-A} x}.$$

Yitang Zhangs Theorem of 2014

There exist constants $\eta, \delta > 0$ such that for any given integer a, we have for

$$\theta(x, q, a) := \sum_{\substack{p \le x \\ p \equiv a \bmod q}} \log p,$$

as $x \to \infty$

$$\sum_{\substack{q \le Q \\ (q,a)=1 \\ q \ y\text{-smooth} \\ q \text{ square-free}}} \left| \theta(x, q, a) - \frac{x}{\varphi(q)} \right| \ll_A \frac{x}{\log^A x}$$

where $Q = x^{\frac{1}{2}+\eta}$ and $y = x^{\delta}$. Zhang used $\eta/2 = \delta = 1/1168$ and, in his application to bounded prime gaps needed $414\eta + 172\delta < 1$.

Why did Zhang extend Bombieri-Vinogradov ?

In the GPY method, to count sums involving $\theta(n+h_i)$, for fixed divisors $d \mid P_{\mathcal{H}}(n)$ we split the sum into arithmetic progressions modulo d and then count using Bombieri-Vinogradov

$$\sum_{j=1}^{k} \sum_{\substack{x<n\le 2x \\ d|P_{\mathcal{H}}(n)}} \theta(n+h_j) = \sum_{j=1}^{k} \sum_{\substack{m \bmod d \\ d|P_{\mathcal{H}}(m)}} \sum_{\substack{x<n\le 2x \\ n\equiv m \bmod d \\ (d,n+h_j)=1}} \theta(n+h_j).$$

To get bounded gaps Zhang needed to push the range of Bombieri-Vinogradov to slightly bigger than $Q=\sqrt{x}$, which is more than half of his paper, and uses highly sophisticated techniques, such as Deligne's mutivariable exponential sum estimates, which come from his celebrated solution to the Riemann hypothesis for varieties over finite fields. Tao/Polymath8a then improved Zhang, replacing **smooth** integers by those they called **densely divisible**, and optimizing his parameters.

What did Maynard and Tao do next?

Figure: James Maynard (1987–) and Terence Tao (1975–)

First, working quite separately, they defined a multivariable weight using

$$d_1 \mid n+h_1, \ldots, d_k \mid n+h_k \text{ instead of } d \mid P_{\mathscr{H}}(n) \text{ and}$$
$$d_1 \cdots d_k \leq R \text{ instead of } d \leq R.$$

Let F be a fixed piecewise differentiable function supported on the simplex

$$\mathscr{R}_k := \left\{ (x_1, \ldots, x_k) \in [0,1]^k : \sum_{i=1}^{k} x_i \leq 1 \right\}.$$

The set of all such functions will be denoted $\mathscr{S}_k$. Maynard optimizes over Fs defined by families of symmetric polynomials of degree two, Polymath of degree three, but also varying the simplex slightly.

The weights

Let W be a product of small primes, $W := \prod_{p \le D_0} p$, $D_0 := \mathrm{logloglog}\, N$, v_0 is chosen so each n has $(n, W) = 1$, i.e. n has no small prime factors. When $(\prod_{i=1}^k d_i, W) = 1$, define the weights:

$$\lambda_{d_1,\ldots,d_k} := \left(\prod_{i=1}^k \mu(d_i) d_i\right) \sum_{\substack{r_1,\ldots,r_k \\ \forall\ i\ d_i | r_i \\ \forall\ i\ (r_i, W)=1}} \frac{\mu(\prod_{i=1}^k r_i)^2}{\prod_{i=1}^k \varphi(r_i)} F\left(\frac{\log r_1}{\log R}, \ldots, \frac{\log r_k}{\log R}\right).$$

When $(\prod_{i=1}^k d_i, W) \neq 1$ set $\lambda_{d_1,\ldots,d_k} = 0$. Maynard offers extensive motivation for this choice of weights, which are fundamental to his method.

Next define two types of integrals depending on F over the simplex:

Let

$$I_k(F) := \int_0^1 \cdots \int_0^1 F(t_1, \ldots, t_k)^2 \, dt_1 \ldots dt_k,$$

$$J_k^{(m)}(F) := \int_0^1 \cdots \int_0^1 \left(\int_0^1 F(t_1, \ldots, t_k) \, dt_m \right)^2 \, dt_1 \ldots dt_{m-1} dt_{m+1} \ldots dt_k.$$

Maynard's fundamental lemma: If F is such that $I_k(F) \neq 0$ and for all m with $1 \leq m \leq k$, $J_k^{(m)}(F) \neq 0$, then $S > 0 \implies \rho + 1$ primes in an $n + \mathscr{H}$, where

$$S_1 = \frac{\varphi(W)^k N \log^k R(1 + o(1))}{W^{k+1}} I_k(F),$$

$$S_2 = \frac{\varphi(W)^k N \log^{k+1} R(1 + o(1))}{W^{k+1}} \sum_{m=1}^{k} J_k^{(m)}(F),$$

$$S := S_2 - \rho S_1 = \sum_{\substack{N \leq n < 2N \\ n \equiv v_0 \bmod W}} \left(\left(\sum_{i=1}^{k} \chi_{\mathbb{P}}(n + h_i) \right) - \rho \right) w_n.$$

What is Polymath?

• Polymath projects were initiated by Timothy Gowers in 2009. Gower's popular Weblog includes "Is massively collaborative mathematics possible?".

• Within a few days he had already formulated and revised a set of rules in "Questions of procedure", which described the process he envisaged.

• He listed several suitable topics and given some initial ideas, in "Background to a Polymath project", to start the process: the Hales-Jowett theorem, the Fursenberg-Katznelson theorem, Szemerédi's regularity lemma, the triangle removal lemma, and so-called sparse regularity lemmas.

• By the time Terry Tao had proposed Polymath8, seven other projects had been initiated.

Is massively collaborative mathematics possible ? (Gowers' Blog)

The idea of a Polymath project is anybody who had anything to say about the problem could contribute brief ideas even if they were undeveloped or maybe wrong.

• Sometimes luck is needed to have the idea that solves a problem. If lots of people think about a problem, then, probabilistically, there is more chance that one of them will get lucky.

• Different people know different things, so the knowledge that a large group can bring to bear on a problem is greater than the knowledge that one or two individuals will have.

• Some folk like to throw out ideas, others to criticize them, others to work out details, others to formulate different but related problems, others to step back from a big muddle of ideas and fashion some more coherent picture out of them, others to compute and construct examples.

• In short, if a large group of mathematicians could connect their brains efficiently, they could perhaps solve problems very efficiently as well.

A computation to ponder: gap to the next prime

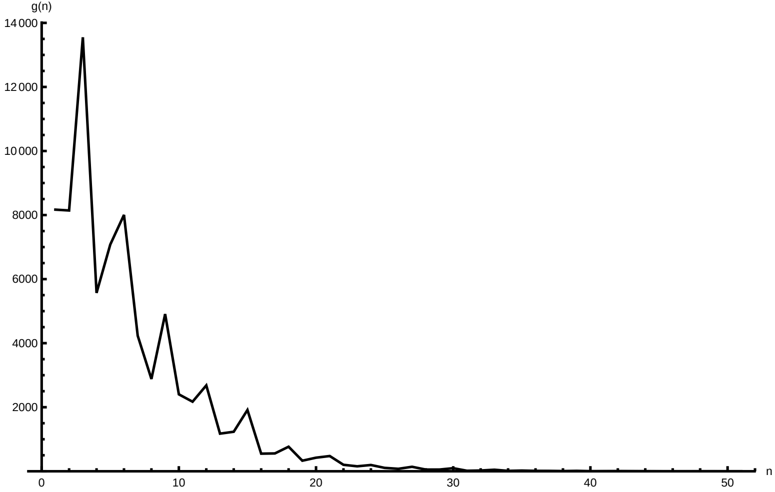

Figure: Next prime gap distribution for primes less than 10^6.

Additional results of Maynard and Polymath

(1) (Maynard) For all $m \in \mathbb{N}$ we have an infinite number of n with $p_{n+m} - p_n \ll m^3 e^{4m} =: B_m$.

(2) (Maynard+Elliot-Halberstam) $p_{n+1} - p_n \leq 12$ for infinitely many n.

(3) (Polymath8b) $p_{n+1} - p_n \leq 246$ for infinitely many n.

(4) (Polymath+Elliot-Halberstam) $p_{n+1} - p_n \leq 6$ for infinitely many n.

Further results are flowing like a stream

Granville's "arithmetic" list:

(1) (Pollack+GRH) Hooley has shown that $GRH \implies Artin$, his primitive root conjecture. Now any such integer g is a primitive root for each of infinitely many m-tuples of primes which differ by no more than B_m.

(2) (Pintz) For all $k \geq 2$, there exists an integer $B > 0$ such that there are infinitely many arithmetic progressions of primes $p_n, \ldots, p_{n+k}$ such that each of $p_n + B, \ldots, p_{n+k} + B$ is also prime.

(3) (Thorner) There exists infinitely many pairs of distinct primes p, q such that both elliptic curves $py^2 = x^3 - x$ and $qy^2 = x^3 - x$ have finitely many (ditto infinitely many) rational points.

(4) (Maynard) For all $x, y \geq 1$ there are $\gg x \exp(-\sqrt{\log x})$ integers $n \in (x, 2x]$ with more than $\gg \log y$ primes in every interval $(n, n + y]$

Ken Ono's students recent result:

(5) (Alweiss, Luo) For any $\delta \in [0.525, 1]$ there exist positive integers k, d such that the interval $[x - x^\delta, x]$ contains $\gg_k x^\delta/(\log x)^k$ pairs of consecutive primes differing by at most d.

Reading list

♦ Zhala films “Counting from infinity”, a movie by George Csicsery.

♠ John Friedlander “Prime numbers: a much needed gap is finally found” (2015).

♠ Kannan Soundararajan “Small gaps between prime numbers: the work of Goldston-Pintz-Yildrim” (2007).

♠ Andrew Granville “Primes in intervals of bounded length” (2015).

♣ Dan Goldston, Janos Pintz, and Cem Yildrim “Primes in tuples I” (2009).

♣ Yitang Zhang “Bounded gaps between primes” (2014).

♣ Polymath8a “New equidistribution estimates of Zhang type” (2014).

♣ James Maynard “Small gaps between primes” (2015).

♣ Polymath8b “Variants of the Selberg sieve, and bounded intervals containing many primes” (2014).

Thanks for listening.