Amazon reviews:

The author has gathered almost 100 year's worth of progress on this family of problems into one volume, and this alone will be very helpful to anyone pursuing research in the field. Recommended.' M. Bona, Choice

'a wonderful tale of how two lesser-known mathematicians worked extremely hard to solve an intriguing, long-standing open problem that so many leading experts could not.' Sam Chow, London Mathematical Society

Amazon reader reviews:

`Helps clarify a difficult subject.
Does a very nice job of guiding the reader through the history of improvements in prime gap estimates, and presents each proof as a series of well-defined steps. Has flow charts showing which results are needed for the important theorems. I find that the motivation for many of the methods in sieve theory are often difficult to understand, and this book really helps. The author's web page has Mathematica software to accompany the book - it could be made more useful with some examples.' T. Dickens

`Very good, understandable and detailed presentation of the topic.' K. Traxel

Popular press rankings:

Book Authority #2 Best prime number book for beginners
Book Authority #28 Best prime number book of all time
LJ Library Journal #12 Academic best sellers in math April 2022

Letter from Yoichi Motohashi 21 February 2021:

Dear Kevin,

I have just received a copy of your impressive book
"Bounded Gaps between Primes". I am deeply
grateful to you for your kind thoughts towards me.

Just by a happy chance, the book came to me on the day of
my 77th birthday party (*).

I am writing a book on elementary and analytic
number theory. I am sure that at its end I shall have the great
pleasure to refer to your book, for a principal aim of my book is
to invite readers to the subject you dealt with thoroughly.

(*) In our tradition, "77th birthday" is an exceptionally merry
and important occasion in one's life. Actually my birthday is 21 Feb;
and my daughter's is 5 Feb. We took the mean value, slightly in favor to me.

All the best,
Yoichi

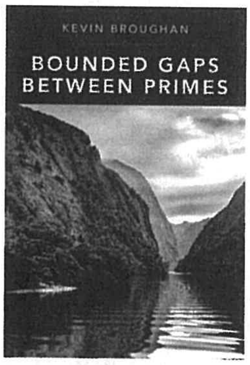
NLMS #506 May 2023

Bounded gaps between primes

by Kevin Broughan, Cambridge University Press, 2021, £40,

ISBN 978-1108799201

Review by Sam Chow



Prime numbers, the building blocks of the integers. We know so much but yet so little about them. By the prime number theorem, the average gap between consecutive primes up to x is roughly $\log x$. This motivates the *Cramér random model*, that a positive integer n has an independent probability $1/(\log n)$ of being prime. Based on this, one might guess that

$$\#\{p_1, p_2 \leq x \text{ primes} : p_1 - p_2 = 2\} \approx \frac{x}{(\log x)^2}.$$

Hardy and Littlewood refined this heuristic by considering divisibility by small primes, and empirical data support their conjecture that

$$\#\{p_1, p_2 \leq x \text{ primes} : p_1 - p_2 = 2\} \sim 2C_2 \frac{x}{(\log x)^2},$$

where

$$C_2 = \prod_{p \neq 2 \text{ prime}} \left(1 - \frac{1}{(p-1)^2}\right) \approx 0.66$$

is the twin prime constant. Even before that, de Polignac had predicted that there are infinitely many twin primes, that is, pairs of primes differing by two—the twin prime conjecture. The twin prime conjecture is one of Landau's four problems presented at the 1912 International Congress of Mathematicians, and is one of the most coveted open problems in number theory.

I first encountered James Maynard at a graduate student conference in Bristol, in May 2013. In his presentation, he described his attempts to prove *bounded gaps between primes*, i.e., that

$$\liminf_{n \to \infty} (p_{n+1} - p_n) < \infty,$$

where $p_1 < p_2 < \ldots$ are the primes. As we were aware, the media had reported just days earlier that unheralded mathematician Yitang Zhang had proved this [4], specifically that

$$\liminf_{n \to \infty} (p_{n+1} - p_n) \leq 70\,000\,000.$$

Maynard was not discouraged, however, and later that year released proofs [2] that

$$\liminf_{n \to \infty} (p_{n+1} - p_n) \leq 600$$

and

$$\liminf_{n \to \infty} (p_{n+m} - p_n) < \infty \qquad (m \in \mathbb{N}).$$

The mathematical community was naturally curious as to the extent to which these bounds could be sharpened using essentially the same methods. The Polymath8 project, led by Terence Tao, was set up for this purpose, with a second goal of understanding and clarifying Zhang's argument. Polymath8a reduced Zhang's bound to 4 680. Polymath8b improved Maynard's bound from 600 to 246, where it remains to this date [3].

Kevin Broughan is an Emeritus Professor at the University of Waikito, who has researched in analytic number theory and is the author of the two-volume work *Equivalents of the Riemann Hypothesis*. In the extensive book under review, he chronicles the full story behind these developments. After an introduction and some background on sieve theory, the book presents early work, the groundbreaking method of Goldston, Pintz and Yildirim [1] upon which subsequent papers build, the aforementioned breakthroughs of Zhang and Maynard, and the meticulous refinements of Polymath8. Computational inputs are also discussed at length, as are related topics.

As well as supplying the mathematical details, Broughan embraces the human aspects of the saga. It is, after all, a wonderful tale of how two lesser-known mathematicians worked extremely hard to solve an intriguing, long-standing open problem that so many leading experts could not. The author draws from many sources, and writes with unbridled passion. He has an unusual but nonetheless effective way of writing proofs, breaking the arguments up into numbered steps, each of which is fairly short. I must caution that the book itself is very long, so the reader would not necessarily want to read every chapter or every proof.

The material is presented at a serious level and is not intended for the general public. *Bounded Gaps Between Primes* is suitable for graduate students in analytic number theory, but others may also find it interesting and informative.

FURTHER READING

[1] D. A. Goldston, J. Pintz and C. Y. Yildirim, Primes in tuples I, Ann. of Math. (2) 170 (2009), 819–862.
[2] J. Maynard, Small gaps between primes, Ann. of Math. (2) 181 (2015), 383–413.
[3] D. J. H. Polymath, The "Bounded Gaps between Primes" Polymath Project: A Retrospective Analysis, EMS Newsletter, December 2014, 13–23.
[4] Y. Zhang, Bounded gaps between primes, Ann. of Math. (2) 179 (2014), 1121–1174.

Sam Chow

Sam Chow is a lecturer of mathematics at the University of Warwick. He is a number theorist working on diophantine approximation and diophantine equations. He enjoys playing chess and other games.

MR4412547 11N05 11-02 11L40 11N36

Broughan, Kevin (NZ-WAIK-NDM)

★**Bounded gaps between primes—the epic breakthroughs of the early twenty-first century.**

Cambridge University Press, Cambridge, 2021. *xiv*+576 *pp.*
ISBN 978-1-108-83674-6; 978-1-108-79920-1

This book is an account of the astonishing progress over the last 20 years or so on establishing the existence of infinitely many closely-spaced prime numbers.

The history of the subject is covered in great detail, leading the reader to the state-of-the-art methods of J. A. Maynard [Ann. of Math. (2) **181** (2015), no. 1, 383–413; MR3272929] and Tao (unpublished) which show that for every m there are infinitely many n such that $p_{n+m} - p_n = O_m(1)$, and the work of the massively-collaborative Polymath group [D. H. J. Polymath, Res. Math. Sci. **1** (2014), Art. 12; MR3373710], which established that $p_{n+1} - p_n \leq 246$ for infinitely many n. Here, p_n denotes the nth prime number.

The book contains detailed elaborations of many lemmas and calculations, and as such its length is rather astonishing at some 576 pages; even then, proofs of standard results such as the zero-free region for $\zeta(s)$, the Siegel-Walfisz theorem and the large sieve inequality are outsourced.

One explanation for the length of the book is the amount of space given to anecdotal remarks and biography, much of which is interesting, though occasionally odd (for example, on Davenport: "One of his better-known sayings is, 'Great mathematics is achieved by solving difficult problems, not by fabricating elaborate theories in search of a problem', which of course is easy to refute") and/or speculative (for example, on Yitang Zhang, "One feels he would prefer many aspects of his quiet mathematical life before 'bounded gaps' ".)

A second reason for the weighty nature of the volume lies in the author's tireless efforts to help the reader navigate the material. For instance, in Chapter 8 alone one finds three full-page proof dependency diagrams.

The book is supported by a linked and freely available package of computer programs, documented in a "mini-manual" from pages 531–554 of the book.

Despite the efforts of the author, the reviewer feels that most people wanting to learn about this topic will prefer to go to the original papers, perhaps supplemented by other expository accounts such as the much shorter book of V. Neale [*Closing the gap*, Oxford Univ. Press, Oxford, 2017; MR3751356] or articles by leaders in the field such as that by A. J. Granville [Bull. Amer. Math. Soc. (N.S.) **52** (2015), no. 2, 171–222; MR3312631], or K. Soundararajan's laudation ["The work of James Maynard", preprint, arXiv:2207.03463] for Maynard's Fields Medal, which was awarded in part for his works discussed in the book.

Ben Joseph Green