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MR3752186 11-02 11M26 11N05 11N25 11N37
Broughan, Kevin [Broughan, Kevin A.] (NZ-WAIK-NDM)
★**Equivalents of the Riemann hypothesis. Vol. 1.**
Arithmetic equivalents.
Encyclopedia of Mathematics and its Applications, 164.
Cambridge University Press, Cambridge, 2017. *xxi*+325 *pp. ISBN* 978-1-107-19704-6

This book is the first of two volumes giving a survey of conjectures equivalent to the Riemann Hypothesis (RH). This volume deals largely with statements of an arithmetic nature, while the second part [K. A. Broughan, *Equivalents of the Riemann hypothesis. Vol. 2*, Encyclopedia Math. Appl., 165, Cambridge Univ. Press, Cambridge, 2017; MR3729257] handles more analytic equivalents.

The book begins with some history, followed by two chapters developing the basic theory. While some proofs are given, other results are quoted from the literature. In addition to many of the classical results, the author presents more recent work in which explicit numerical constants appear. These lead to the proof in Chapter 4 of Schoenfeld's criterion, that the RH is equivalent to the statement that

$$|\psi(x)-x| \le \frac{\sqrt{x}(\log x)^2}{8\pi}, \quad \text{for } x \ge 74.$$

Next come various other arithmetic statements, beginning with a result of Nicolas, that RH holds if and only if

$$\frac{N_k}{\phi(N_k)} > e^\gamma \log\log N_k \quad \text{for every } k \ge 1,$$

where N_k is the product of the first k primes. Chapters 6 and 7 then lead up to Robin's criterion, that RH holds if and only if

$$\sigma(n) < e^\gamma n \log\log n \quad \text{for all } n \ge 5041.$$

This is perhaps the central topic for this volume, and there is a comprehensive discussion of various kinds of record-breaking, abundant numbers.

Chapter 10 contains a variety of sundry criteria, including the Franel criterion (concerning the distribution of Farey fractions) and Redheffer's criterion, requiring $\det(R_n) \ll_\varepsilon n^{1/2+\varepsilon}$, where $R=(r_{ij})$ is the $n\times n$ matrix with $r_{ij}=1$ if $j=1$ or $i \mid j$, and $r_{ij}=0$ otherwise.

The book ends with two appendices, the first giving some useful tables and the second describing the author's software package, which is available to readers wishing to explore numerically some of the above material. Finally there is a bibliography of nearly 200 items.

The book does not suggest that RH might be proved via any of these criteria. Instead its intention is to show the reader the many different subject areas that are connected to the problem, and thereby motivate the reader to learn more about them. The exposition should be suitable for strong undergraduates; but experienced researchers will also find something here to interest them. *D. R. Heath-Brown*

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Citations | From References: 4 | From Reviews: 1

MR3729257 11M26 11N05 11N25 30D10 30D15 33C45
Broughan, Kevin [Broughan, Kevin A.] (NZ-WAIK-MS)
★Equivalents of the Riemann hypothesis. Vol. 2.
Analytic equivalents.
Encyclopedia of Mathematics and its Applications, 165.
Cambridge University Press, Cambridge, 2017. *xx*+491 *pp.*
ISBN 978-1-107-19712-1; 978-1-108-29078-4

This book is the second half of the author's survey of conjectures equivalent to the Riemann Hypothesis, and is devoted to statements of a largely analytic, rather than arithmetic, nature. The book does not suggest that any of these equivalents provides a useful line of attack on the Riemann Hypothesis. However, it clearly demonstrates that by studying the many diverse equivalent statements that have been established one can encounter an impressive spectrum of mathematical topics. It unquestionably showcases the range of subject areas on which the zeta-function impinges.

The book begins with the criteria of Riesz and of Hardy and Littlewood, involving the power series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{(n-1)!\zeta(2n)} x^n \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{(-1)^n}{n!\zeta(2n+1)} x^n$$

respectively. Next comes the Nyman criterion, featuring the functions

$$f_\theta(x) = [\theta/x] - \theta[1/x],$$

for $0 < \theta \leq 1$ as elements of $L^2(0,1)$. Subsequent chapters discuss Li's criterion, the De Bruijn–Newman constant, the approximation of $\xi(\frac{1}{2} + iz)$ by orthogonal polynomials, and the height of products of cyclotomic polynomials. The chapter on integral equations gives the elegant condition

$$\int_{-\infty}^{\infty} \frac{\log|\zeta(\frac{1}{2} + it)|}{1 + 4t^2} dt = 0,$$

which was shown to be equivalent to the Riemann hypothesis by Balazard, Saias and Yor.

Chapter 9 is somewhat different, discussing Weil's version of the explicit formula, with the resulting positivity condition for the Riemann hypothesis. The chapter goes on to present a proof of the Riemann hypothesis for elliptic curves over finite fields, modulo some background reading.

There is also a chapter on Dirichlet L-functions and the generalized Riemann hypothesis. This exposes the reader to further arithmetic connections, and the book develops some of these, including the Bombieri-Vinogradov theorem, for example.

The book concludes with several substantial appendices, giving further background material. There is also information about the author's software package available to readers wishing to explore numerically some of the topics from the book.

Throughout the book careful proofs are given for all the results discussed, introducing an impressive range of mathematical tools. Indeed, the main achievement of the work is the way in which it demonstrates how all these diverse subject areas can be brought to bear on the Riemann hypothesis.

The exposition is accessible to strong undergraduates, but even specialists will find material here to interest them.

{For Vol. 1 see [K. A. Broughan, *Equivalents of the Riemann hypothesis. Vol. 1*, Encyclopedia Math. Appl., 164, Cambridge Univ. Press, Cambridge, 2017; MR3752186].}

D. R. Heath-Brown

"Niew Archief voor Wiskunde"
by Pieter Moree

geschreven door twee auteurs. Artin schrijft zelf het gedeelte dat de hoofdstelling bevat en laat het schrijven van de toepassing over aan Arthur Milgram. De formules van Cardano of Ferrari vindt men niet in dit boek. Afhankelijk van de druk zult u zien dat de voorbeelden in dit boek op één hand te tellen zijn en dat het geen enkele opgave bevat. Dat het boek *Galois Theory Through Exercises* van Brzeziński het andere uiterste is, doet de titel al vermoeden. Er is overigens ook een relatie met het boek van Artin. In het voorwoord schrijft Brzeziński dat hij in de jaren zestig van de vorige eeuw al colleges gaf en dat men toen noodgedwongen vaak gebruikmaakte van het boek van Artin of een gedeelte uit het bekende algebraboek van Van der Waerden. Beide boeken blinken niet uit door voorbeelden en opgaven. Brzeziński heeft destijds voor zijn studenten sets van opgaven met uitwerkingen gemaakt en deze presenteert hij nu samen met de Galoistheorie in boekvorm. Een prima idee.

Het boek van Brzeziński bestaat uit negentien hoofdstukken en een appendix. Het appendix bevat de benodigde basiskennis. De eerste vijftien hoofdstukken behandelen de volgende onderwerpen: het oplossen van algebraïsche vergelijkingen, uitbreidingen van lichamen, polynomen en irreducibiliteit, algebraïsche uitbreidingen, splijtlichamen, automorfismengroepen van lichamen, normale uitbreidingen, separabele uitbreidingen, Galoisuitbreidingen, cyclotomische uitbreidingen, Galoismodules, oplosbare groepen, oplosbaarheid van vergelijkingen, meetkundige constructies en het berekenen van Galoisgroepen. Dit zijn de onderwerpen die men tegenwoordig in de meeste boeken over Galoistheorie aantreft.

Kenmerkend voor de eerste vijftien hoofdstukken is dat die elk uit twee delen bestaan: het eerste deel bevat definities en stellingen *zonder* bewijzen en het tweede deel bestaat uit opgaven waaronder ook een gering aantal opgaven met een softwarepakket. Het totaal der opgaven is nagenoeg 200.

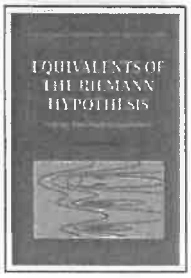
Hoofdstuk 16 bevat nog eens circa 100 extra opgaven met soms een aanwijzing of een verwijzing. De bewijzen van alle stellingen vindt men in hoofdstuk 17. De oplossingen van de opgaven vindt men in de hoofdstukken 18 en 19; in hoofdstuk 18 staan aanwijzingen en antwoorden en in hoofdstuk 19 voorbeelden en geselecteerde oplossingen. Voor elke opgave moet men beide hoofdstukken raadplegen. Het zal wel duidelijk zijn dat men bij het doorwerken van het boek veelvuldig bladert van het ene naar het andere hoofdstuk. Ik denk dat het boek *rustiger* was geworden als de bewijzen van de stellingen gewoon bij elk hoofdstuk stonden en als de oplossingen van de problemen niet gesplitst waren over twee aparte hoofdstukken. Op sommige plekken in het boek kan men ook gebruikmaken van het softwarepakket Maple; het is maar een heel klein gedeelte van het boek. Ik denk dat het beter was geweest het boek niet te verbinden met een specifiek softwarepakket omdat het dan het risico loopt snel te verouderen op dat punt en dat is in dit geval zeker jammer omdat de rest als het ware eeuwigheidswaarde heeft.

Ik heb enkel kritiek op de vorm van het boek; op de inhoud valt niets af te dingen. Alles is zeer goed te volgen en uitgewerkt. Het bestuderen van de Galoistheorie in wisselwerking met veel voorbeelden en opgaven met uitwerkingen (de oplossingen van de oefeningen beslaan ongeveer tachtig pagina's, een derde van het boek zonder het appendix) is zeer nuttig en voor zo'n aanpak is dit boek zeker geschikt. *Math Dicker*

Kevin Broughan

Equivalents of the Riemann Hypothesis
Volume 1: Arithmetic Equivalents

Cambridge University Press, 2017
xx + 325 p., prijs £99.99
ISBN 9781107197046



Kevin Broughan

Equivalents of the Riemann Hypothesis
Volume 2: Analytic Equivalents

Cambridge University Press, 2017
xix + 491 p., prijs £120.00
ISBN 9781107197121

These two volumes give a survey of conjectures equivalent to the Riemann Hypothesis (RH). The first volume deals largely with statements of an arithmetic nature, while the second part considers more analytic equivalents.

The Riemann zeta function, is defined by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \tag{1}$$

with $s = \sigma + it$ a complex number having real part $\sigma > 1$. It is easily seen to converge for such s. By analytic continuation the Riemann zeta function can be uniquely defined for all $s \neq 1$. In $s = 1$ it has a simple pole. In 1859 Bernhard Riemann published 'Über die Anzahl der Primzahlen unter einer gegebenen Grösse'. This only 9-pages-long paper, the only published work of Riemann on number theory, is without doubt the most important paper ever written in analytic number theory; indeed it is foundational, as Riemann makes essential use of s being a complex variable (allowing methods of complex analysis to be applied), whereas a century earlier Leonhard Euler only considered $\zeta(s)$ for real values of s.

The uniqueness of prime factorization of the integers finds its analytic counterpart in the identity $\zeta(s) = \prod_p (1-p^{-s})^{-1}$ valid for $\sigma > 1$ (where the product extends over all the primes p). Given this identity it is perhaps not so surprising that the behaviour of $\zeta(s)$ is very closely related to the distributional properties of the primes. The Riemann Hypothesis (formulated in the 1859 paper) states that all the zeros in the critical strip $0 < \sigma < 1$ are on the line $\sigma = \frac{1}{2}$. If true, it implies that the prime counting function $\pi(x)$, that counts the primes $p \leq x$, behaves in a fairly regular way. Indeed, Helge von Koch proved in 1901 that the RH is equivalent with $\pi(x) = \int_2^x du/\log u + O(\sqrt{x}\log x)$. The celebrated prime number theorem says that $\pi(x)$ asymptotically behaves as $x/\log x$. That is a much weaker statement and is equivalent with there being no zeta zeros on the line $\sigma = 1$. That there are no zeros with $\sigma > 1$ is a consequence of the prime product identity for $\zeta(s)$.

It would go too far here to discuss all chapters and I will limit myself to some chapters that are either close to my mathematical expertise or those discussing some of the most famous RH equivalences. Most of the criteria have their own chapter devoted to them, Chapter 10 has various criteria that are discussed more briefly. A nice example is Redheffer's criterion. It states that RH holds true if and only if for every $\epsilon > 0$, we have $\det(R_n) \le C_\epsilon n^{1/2+\epsilon}$, where $R_n = (r_{ij})$ is the $n \times n$ matrix with $r_{ij} = 1$ if $j = 1$ or i divides j, and $r_{ij} = 0$ otherwise.

In Chapter 4, after some chapters on history, basic properties of $\zeta(s)$ and one with derivations of some basic estimates involving prime numbers, Schoenfeld's criterion is proved. It says that the RH is equivalent to the inequality that $|\psi(x) - x| \le \sqrt{x}(\log x)^2/(8\pi)$ for $x \ge 74$, where $\psi(x) = \sum_{p^n \le x} \log p$. The sum is over all prime powers $p^n \le x$ and each of those chips in a weight $\log p$. The function $\psi(x)$ turns out to be easier to study than $\pi(x)$, on the other hand usually results on $\psi(x)$ can be easily translated to results on $\pi(x)$.

Schoenfeld's criterion is important, but not very surprising. More surprising is Robin's criterion which states that, for $n > 5040$,

$$\sum_{d \mid n} \frac{1}{d} < e^\gamma \log\log n, \tag{2}$$

if and only if the Riemann Hypothesis holds true (where γ denotes Euler's constant). Traditionally (2) is written as $\sigma(n) < e^\gamma n \log\log n$, where $\sigma(n) = \sum_{d \mid n} d$ denotes the sum of divisors of n. The author speaks of the Ramanujan–Robin criterion as Ramanujan proved that (2) holds, under RH, for all n sufficiently large. Chapter 8 opens with a discussion of a paper on this criterion, where we (Choie, Lichiardopol, the reviewer and Solé) establish that the inequality holds for a large class of integers. We showed, for example, that all odd integers > 9 satisfy (2) (thus to wit: we solved half of RH!). In addition we showed that all 5-free integers > 5040 satisfy (2), where an integer is said to be k-free if no k^{th} power of an integer > 1 divides it (otherwise it is said to be k-full). It then continues to discuss various improvements of our work (the author and Trudgian showed for example that the inequality holds for 11-free integers). Not discussed, but fresh on the arXiv is the result of Morrill and Platt that RH is true if and only if (2) holds for all 20-full integers.

We say that N is a colossally abundant number if for some $\epsilon > 0$ the function $\sigma(n)/n^{1+\epsilon}$ attains its maximum at N. It is not difficult to see that if a counterexample to (2) exists for some $n > 5040$, there exists a colossally abundant counterexample. Chapter 9 is devoted to a study of these numbers and some variations.

An inequality closely related to that of Robin is

$$\frac{n}{\varphi(n)} < e^\gamma \log\log n. \tag{3}$$

This is now called the Nicolas inequality. The intriguing work of mostly Nicolas related to his inequality and refinements thereof are discussed in Chapter 5.

About five chapters in total are devoted to this and closely related material, thus making it one of the main topics discussed in Volume 1.

I will now discuss Volume 2, which draws on a rather broader spectrum of mathematical methods and ideas than Volume 1.

Marcel Riesz showed in 1916 that RH is equivalent to

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(n-1)!\zeta(2n)} x^n \le C_\epsilon x^{1/4+\epsilon}.$$

This implies that the values of $\zeta(s)$ at all even integers determine the truth of RH. Hardy and Littlewood proved a variation where all the values of $\zeta(s)$ at odd integers > 1 appear. In 2005 Luis Báez-Duarte unified and generalized the existing series equivalences, giving rise to a very large family. Chapter 2 provides the details.

One of the most beautiful and surprising equivalences to the Riemann Hypothesis is related to Banach and Hilbert space methods. By $\lfloor r \rfloor$ we denote the entire part of a real number r. Bertil Nyman in 1950 in his PhD thesis (!) proved that RH is true if and only if the linear span of the functions $k_\alpha(x) := \lfloor \alpha/x \rfloor - \alpha\lfloor 1/x \rfloor$ is dense in the Hilbert space $L^2(0,1)$ of square integrable functions on the interval $(0,1)$, where $0 < \alpha \le 1$. Arne Beurling, the PhD supervisor of Nyman, extended the criterion to L^p-spaces. These intriguing results are discussed in Chapter 3.

Put

$$\lambda_k := \sum_\rho \left(1 - \left(1 - \frac{1}{\rho}\right)^k\right),$$

where the sum is over the non-trivial Riemann zeta zeros. Li's celebrated criterion says that RH is true if and only if $\lambda_k > 0$ for every $k \ge 1$. Note that $\lambda_1 = \sum_\rho 1/\rho$. It was already known to Riemann that $\lambda_1 = 1 + \gamma/2 - \log(4\pi)/2$. Li's result inspired a lot of follow up work.

The Riemann zeta function after multiplication by some simple factors can be made real on the line $\sigma = \frac{1}{2}$. This function can be written as the Fourier transform $\int_{-\infty}^{\infty} \Phi(u) e^{izu} du$, with $\Phi(u)$ completely explicit. George Pólya considered the family of deformations

$$\Theta_\lambda(z) := \int_{-\infty}^{\infty} e^{\lambda u^2} \Phi(u) e^{izu} du.$$

RH is equivalent with $\Theta_0(z)$ having only real zeros. It can be shown there is a unique finite real number Λ such that $\Theta_\lambda(z)$ has only real zeros if and only if $\lambda \ge \Lambda$. The constant Λ is now called the de Bruijn–Newman constant. RH is equivalent with $\Lambda \le 0$. De Bruijn showed in 1950 that $\Lambda \le \frac{1}{2}$. Newman showed that Λ exists, i.e. that $\Lambda > -\infty$. The lower bound was improved many times over the years and the best we know currently is that $-10^{-11} < \Lambda < \frac{1}{2}$.

Chapter 6 concerns a criterion of 2006 from Cardon and Robert which in essence is about approximation of $\zeta(\frac{1}{2} + iz)$ by a particular sequence of orthogonal polynomials.

Amoroso showed that if $A_N(x) = \prod_{n \le N} \Phi_n(x)$ is the product of the first N cyclotomic polynomials, then its so-called height, for any $\epsilon > 0$, is bounded above by $C_\epsilon N^{\lambda+\epsilon}$ (with C_ϵ a constant) if and only if $\zeta(s)$ has no zero with real part exceeding λ. This result is proved in Chapter 7.

Chapter 9 concerns Weil's version of the explicit formula. For a large class of test functions this relates a sum involving Riemann zeros to a sum involving primes and some remaining terms that can be regarded as associated with the so-called prime at infinity. By a judicious choice of test functions this allows one to prove many results.

Weil's work on explicit formulae has been very influential and also was an input for a proof of the RH for curves over finite fields (in the case of elliptic curves a proof is given here). It has been the goal of many to use this approach for RH and generalizations thereof. Even though Weil provided a bridge between the two cases and meanwhile vast and deep new mathematical theories have been developed, Weil's bridge remains to be crossed...

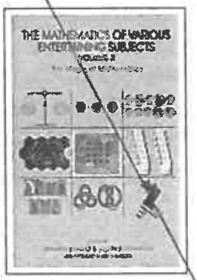
The book concludes with several appendices, giving more background material, for example on the Fourier, Laplace and Mellin transform. There is also a manual for a set of functions written to assist the reader to reproduce and possibly extend, calculations mentioned in the book.

The first volume falls mostly in the realm of computational number theory and the author rederived some results, for example some of the classical ones of Rosser and Schoenfeld. The second volume draws on much more areas of mathematics and everybody, I think, will be in for some surprises there. For the results stated mostly proofs are given. This requires a broad range of mathematical tools to be used. Sometimes the author cleaned up some of the original proofs.

All in all these books serve as a good introduction to a wide range of mathematics related to the Riemann Hypothesis and make for a valuable contribution to the literature. They are truly encyclopedic and I am sure will entice many a reader to consult some literature quoted and who knows, eventually make an own contribution to the area.

I thank Alexandre Kosyak for helpful comments on several earlier versions of this review.

Pieter Moree



Jennifer Beineke, Jason Rosenhouse (eds.)

The Mathematics of Various Entertaining Subjects

Volume 3: The Magic of Mathematics

Princeton University Press, 2019
xxi + 325 p., prijs $49.95
ISBN 9780691182582

Dat recreatieve wiskunde heel vaak aan de basis heeft gestaan van belangrijke wiskundige doorbraken zal bij een ieder van u inmiddels wel bekend zijn. Minder bekend (althans bij mij) is dat er sinds 2013 een tweejaarlijkse MOVES (Mathematics of Various Entertaining Subjects) Conference wordt georganiseerd door het MOMATH (Museum of Mathematics) dat sinds 2012 zijn deuren heeft geopend in New York. Ook van de meest recente MOVES-conferentie (in 2017 georganiseerd, met het thema The Magic of Mathematics) is nu dus een boek verschenen (net als de delen 1 en 2 samengesteld door math-professors Beineke en Rosenhouse) waar de afwezige alsnog (en de aanwezige wederom) zijn/haar hart kan ophalen aan in dit geval 18 artikelen (respectievelijk zes, vijf, vier en drie in de vier hoofdcategorieën 'Puzzles and Brainteasers', 'Games', 'Algebra and Number Theory' en 'Geometry and Topology'), waarbij een artikel uiteraard telkens uitvoerig een onderwerp behandelt in een van de genoemde categorieën. Soms is een artikel niets anders dan een serie problemen, gelukkig met oplossingen die — nog gelukkiger — vrijwel direct geplaatst staan achter de problemen, dus geen nodeloos geblader de hele tijd, en onder andere dat maakt dit tot een prettig, helder opgebouwd en sowieso zeer leesbaar boek.

Een eerste voorbeeld van een intrigerend probleem met een verrassende oplossing (zie verder) — uit de deelcategorie 'Probability in Your Head' — is het probleem 'Random Rice'. In de plaatselijke rijstwinkel staat een machine die door de knop een keer in te drukken telkens een willekeurige (uniform verdeelde) hoeveelheid rijst tussen 0 en 1 beker levert. Hoe vaak gemiddeld moet je de knop indrukken om in totaal tenminste 1 volle beker rijst te hebben? Het antwoord hierop is (uiteraard) niet het wellicht verwachte twee keer.

Minstens zo intrigerend is het probleem er direct achter ('Six with No Odds'), namelijk hoe vaak moet je met een gewone dobbelsteen gooien tot je een 6 gooit, als je op weg naar die 6 nooit een oneven aantal ogen gooit? Even verrassend is het weer dat het antwoord niet het verwachte aantal 3 is.

Verrast te worden is prettig in elk boek — zeker in een boek als dit — en gelukkig gebeurt dat heel vaak. Inherent aan de gekozen opzet is de kans al vrij klein dat je van het onderwerp al wat af weet (want vaak geheel nieuw), laat staan dat je een in een artikel genoemd probleem zelf al zou kennen. Maar zelfs als je een liefhebber of zelfs een kenner zou zijn van bijvoorbeeld balansproblemen dan kom je (in de deelcategorie 'Coins and Logic'; lees ook de boeken van de uiteraard ook in het boek genoemde grootmeester van de logica Raymond Smullyan) toch vaak weer iets tegen wat je nog niet kende. Voor dit (derde) probleem definiëren we allereerst een *reverse scale* (als tegenhanger van een *true scale*) als een weegschaal die bij twee gelijke gewichten in balans is maar bij twee verschillende gewichten de lichtste aanwijst als zwaarste en andersom (heel eenvoudig te detecteren door 1 munt op een van de twee schaaltjes te leggen). Nu het probleem. Er zijn drie identiek uitziende en even zware weegschalen die *true* of *reverse* kunnen zijn. Je hebt de beschikking over minstens drie munten en precies één daarvan is vals en lichter dan de andere munten. Wat is het kleinste aantal wegingen dat nodig is om erachter te komen van welk type (*true* of *reverse*) elke weegschaal is? Ook hier weer een uiterst verrassend antwoord en een nog verrassender redenering.

Ook al zijn sommige artikelen wat theoretischer van aard, altijd wordt de lezer (net als de toeschouwer bij de oorspronkelijke presentaties mag ik aannemen) meegenomen door het geven van geschikte instap- of tussenproblemen en het bewijzen van toepasselijke lemma's. Verder is de benodigde wiskundige kennis op een enkel uitstapje na maximaal die van een eerstejaarsstudent wiskunde.

Ik zal de verleiding moeten weerstaan een samenvatting te geven van of interessante problemen te citeren uit elk van de 18 artikelen, maar ik denk dat het noemen van nog wat meer onderwerpen die in dit boek de revue passeren meer dan genoeg uw interesse kan opwekken en dat dit fraaie boek heel veel denk- en leesplezier zal genereren. Welnu, mocht u iets meer (en vooral iets nieuws) willen weten over flexagons, Japanse KenKen-puzzels, bingo-paradoxen, het getal 142857, de spel-app Khalou, het kortste route-probleem in een driehoekig rooster, partities van kwadraten in positieve natuurlijke getallen waarvan de omgekeerden som 1 hebben, balansproblemen en LEGO-bouwwerken en nog veel meer en bovendien over een aantal meer dan verbluffende kaarttrucs wilt beschikken (en tevens iets wilt weten over de (kaart)magie van Charles Sanders Peirce, één van de grootste zo niet de grootste van alle Amerikaanse logici), dan is dit het boek. Ongetwijfeld zijn de mij niet bekende eerder verschenen delen (Volume 1 en Volume 2) op dezelfde manier samengesteld en even onderhoudend als dit derde deel in een serie die wat mij betreft nog heel lang mag doorlopen. Wilt u het fijne weten van het hoe en waarom van de antwoorden op de drie hierboven geformuleerde problemen (respectievelijk e, $1\frac{1}{2}$ en 1), dan zult u tot aanschaf moeten overgaan.

Joop van der Vaart

the terminology that is vital to understanding our discipline.

Experienced mathematicians may not learn any new mathematical ideas, but this book will certainly help all to appreciate the different ways in which others think about them. Going beyond the content, the style of the book encourages you to not only reflect upon your own mathematical understanding of fundamental topics, but also the way in which we might communicate them. Although those currently teaching mathematics were not listed amongst its intended audience, I am sure that they too will find something of value here.

Michael Grove

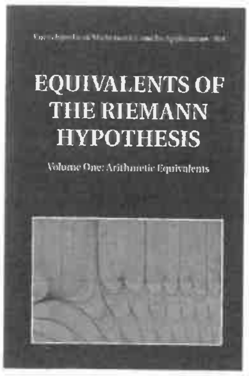
Michael Grove is a Reader within the School of Mathematics at the University of Birmingham. He is also Honorary Secretary of the Institute of Mathematics and its Applications with responsibility for education. Michael has a number of hobbies that seem to depend upon the time of year, most notably DIY, playing the guitar, and the latest being an attempt to finally master the game of golf.

Newsletter of the LMS 480 Jan. (2019)

Equivalents of the Riemann Hypothesis, volumes one and two

by Kevin Broughan, Cambridge University Press, 2017, £195, US$ 250, ISBN: 978-1-108-29078-4

Review by R.S. MacKay



Riemann's hypothesis is considered by many to be the most important open problem in mathematics. It has resisted 150 years of attempts by a large number of fine minds to either prove or disprove it.

In these two volumes, Kevin Broughan surveys in detail a wide range of equivalents of Riemann's hypothesis.

I state the Riemann hypothesis in the form given by Riemann himself, namely that all zeroes of the entire function

$$\xi(t) = 4\int_1^{\infty} \frac{d(x^{\frac{3}{2}}\psi'(x))}{dx} x^{-\frac{1}{4}} \cos(\frac{t}{2}\log x)\, dx$$

are real, where Jacobi's $\psi(x) = \sum_{n=1}^{\infty} e^{-n^2\pi x}$. Most subsequent authors changed Riemann's notation by writing ξ as a function of $s = 1/2 + it$. Broughan continues the latter convention but in this review I translate to Riemann's form.

The fundamental significance of the Riemann hypothesis is for the distribution of primes, namely Riemann's hypothesis is equivalent to $\pi(x) - li(x) = O(\sqrt{x}\log x)$, where $\pi(x)$ is the number of primes less than x and the logarithmic integral $li(x)$ is the principal value of $\int_0^x \frac{dt}{\log t}$.

In the quest for proofs or disproofs of the Riemann hypothesis, and hence of the above statement, many people have come up with other equivalents. Such equivalents allow you to keep an open mind about whether you want to try to prove Riemann's hypothesis true or false. They will also allow you to deduce many things as soon as you prove it or disprove it.

The first volume surveys the principal arithmetic equivalents of the Riemann hypothesis. They are weird and wonderful. Here are some examples:

Schoenfeld $|\psi(x) - x| \leq \frac{\sqrt{x}\log^2 x}{8\pi}$ $\forall x \geq 74$, where Chebyshev's $\psi(x) = \sum_{p^m \leq x} \log p$;

Littlewood $\sum_{n \leq x} \mu(n) \ll x^{\frac{1}{2}+\varepsilon}$ $\forall \varepsilon > 0$, where Möbius' $\mu(n)$ is 0 if n has a repeated prime factor, else ±1 according as the number of prime factors is even or odd;

Robin $\sigma(n) < e^{\gamma} n \log\log n$ $\forall n \geq 5041$, where $\sigma(n)$ is the sum of the divisors of n and $\gamma = \lim_{n\to\infty} \sum_{m=1}^{n} 1/m - \log n$;

Nicolas $\frac{ne^{-\gamma}}{\varphi(n)} < \log\log n + \frac{4+\gamma-\log 4\pi}{\sqrt{\log n}}$ $\forall n \geq N_{120569}$, where Euler's $\varphi(n)$ is the number of integers in $[1, n]$ coprime to n, and N_k is the product of the first k primes.

Perhaps the most delightful is the Caveney-Nicolas-Sondow equivalent: the only extraordinary number is 4 (where "extraordinary" is defined in terms of abundance of divisors).

The second volume surveys many analytic equivalents of the Riemann hypothesis. It begins with the Riesz and Hardy-Littlewood equivalents, about growth of certain power series with coefficients involving the zeta function at even or odd positive integers respectively. Next is the Nyman-Beurling criterion about certain subsets of a Banach space having dense span. This is followed by Lagarias' criterion that the logarithmic derivative of Riemann's ξ has positive imaginary part in the lower half plane, the Sondow-Dumitrescu criterion that $|\xi|$ is strictly increasing along vertical lines in the upper halfplane, and Li's criterion about the positivity of certain sums over the zeroes of ξ. Many other equivalents are treated, though the volume is not encyclopaedic. The last one I'll describe is the de Bruijn-Newman criterion. This is a tautology in my opinion but interesting nonetheless. Riemann's ξ can be considered as ξ_0 in a family ξ_λ of functions such that $\partial\xi/\partial\lambda = \partial^2\xi/\partial t^2$. Diffusion never creates real zeroes and it removes multiple ones. There is a $\Lambda \in \mathbb{R}$ such that for $\lambda \leq \Lambda$ all zeroes are real and for $\lambda > \Lambda$ there are some complex ones. So Riemann's hypothesis is equivalent to $\Lambda \geq 0$. Newman conjectured that $\Lambda \leq 0$ and this has recently been proved by Rodgers and Tao [4] (NB: who use the opposite sign convention). So the Riemann hypothesis is equivalent to $\Lambda = 0$, and if the Riemann hypothesis is true it is on the edge of being true, which may explain why it is so difficult to settle.

Broughan's two volumes are not an introductory text, so the uninitiated would be well advised to first read something that starts from the beginning. For example, alternative surveys of equivalents of the Riemann hypothesis, with introductory material, appear in Conrey [3], Chapter 5 of Borwein et al. [2], and Section 4 of Balazard [1], but they are nowhere near as comprehensive as Broughan's two volumes. He gives proofs for most of the stated results.

Nonetheless, Broughan does summarise basic material, making the two-volume set self-contained, and with a minimum of cross-referencing between them. He has also written Mathematica scripts to evaluate many relevant functions, available on his website, which allow one to play around and reproduce various steps.

The two volumes are a very valuable resource and a fascinating read about a most intriguing problem.

FURTHER READING

[1] M. Balazard, Un siècle et demi de recherches sur l'hypothèse de Riemann, Soc Math France Gazette 126 (2010) 7–24.

[2] P. Borwein, S. Choi , B. Rooney, A. Weirathmueller, The Riemann Hypothesis, Springer, 2008.

[3] J.B. Conrey, The Riemann Hypothesis, Notices Am. Math. Soc. 50 (2003) 341–353.

[4] B. Rodgers, T. Tao, The de Bruijn-Newman constant is non-negative, arXiv: 1801.05914 (2018).

Robert MacKay

Robert MacKay is a Professor in Mathematics at the University of Warwick and Director of Mathematical Interdisciplinary Research. His main research interests are in dynamical systems and their applications, but he developed an obsession with Riemann's hypothesis, starting on New Year's day 2015. He used to fantasise about becoming a rock star but settled for a safer life as an academic.

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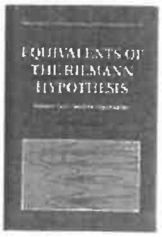
Equivalents of The Riemann Hypothesis, Volume Two: Analytic Equivalents



Kevin Broughan

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[Reviewed by Steven Deckelman, on 01/7/2019]

The Riemann Hypothesis has aptly been described as the holy grail of mathematics. Like a medieval dragon that smote all heros who dared try slaying it, the proof or disproof of the Riemann Hypothesis has consistently withstood the efforts of the best mathematical minds over the past century and a half. This includes, incidentally, Riemann himself, who neither offered a proof nor gave any indication he had one.

The Riemann Hypothesis is a remarkable assertion about two seemingly very disparate mathematical objects, the distribution of prime numbers and the location of zeros of a specific analytic (actually meromorphic) function. Originally conjectured by Riemann in 1859 in a short note to the Prussian Academy that contained no proofs, this problem has proved to be a particulary singular and recalcitrant adversary. Mathematical problems like this are important not only in themselves and the problems they would settle upon their resolution, but also because of the mathematics that is created through the attempts of many mathematicians over the years to understand them. One way to see this is to examine the myriad techniques that have been developed connecting it to other areas of mathematics through its various equivalents. In the case of the Riemann Hypothesis the connections are astonishing and include such diverse areas as number theory (of course), numerical analysis, graph theory, group theory, matrix theory, functional analysis (Banach and Hilbert spaces), orthogonal polynomials, Hermitian forms, discrete measures and even quantum mechanics (Hilbert-Pólya conjecture).

There are many good books on the Riemann hypothesis, so this review will not dwell on the origins of the conjecture except to say that its connection to the distribution of primes first arises in the Euler product formula

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p} (1 - p^{-s})^{-1},$$

initially defined for $s \in \mathbb{C}$ in the half-plane $Re(s) > 1$ but then extended by analytic continuation on the whole complex plane except at $s = 1$ where it has a simple pole of residue 1. The existence of the pole at $s = 1$ implies that the number of primes are infinite. It is easy to see that ζ has simple zeros at the negative even integers and has been known for some time (proved first by Hardy) that it has infinitely many zeros on the critical line $\mathrm{Re}(s) = \frac{1}{2}$. The Riemann Hypothesis is, of course, that apart from the trivial zeros at the negative even integers, all of its nontrivial zeros fall on this critical line and no nontrivial zero off this critical line has ever been found.

More generally there are various generalizations such as the "Riemann Hypothesis" for Dirichlet L-functions

$$L(s, \chi) = \sum \frac{\chi(n)}{n^s}$$

where χ is a Dirichlet character. The classical zeta function corresponds to $\chi(n) = 1$.

This two volume catalogue of many of the various equivalents of the Riemann Hypothesis by Kevin Broughan is a valuable addition to the literature. Its intended audience include graduate students and researchers in number theory, though most of it is quite accessible to non-specialists, the main prerequisite being a graduate course in analysis, especially complex analysis. The important ideas are summarized in several appendices. But even some of the more advanced topics, e.g. the Weil conjectures in chapter 9 of Volume Two would be good starting points for those wanting to learn more about these topics.

Volume One begins with a chapter on the history of the Riemann Hypothesis and its derivation and is mostly classical in flavor. It is mostly concerned with various equivalents that have been given involving arithmetic functions, for example the prime counting function

$$\pi(x) = \sum_{p \leq x} 1$$

and the Chebyshev function

$$\psi(x) = \sum_{p \leq x} \left\lfloor \frac{\log x}{\log p} \right\rfloor \log p.$$

One of the early equivalences of this type, proved by Von Koch in 1901 is that the Riemann hypothesis is equivalent to

$$\pi(x) = \mathrm{Li}(x) + O(\sqrt{x}\log(x)) \text{ as } x \to \infty$$

where

$$\mathrm{Li}(x) = \int_0^\infty \frac{dx}{\log x}.$$

These often take the form of an inequality asserted for sufficiently large x , for example, the Riemann hypothesis is also equivalent to

$$\psi(x) = x + O\left(x^{\frac{1}{2}} \log^2 x\right) \text{ as } x \to \infty.$$

The proofs of these two equivalences, as well as a great many others, are given in the book. When a proof is not included references are given where to find it. While these two equivalents have clear connection to the primes, there are others whose connection is not as immediately evident. One of my favorites is the symmetric group criterion, discovered only in the late 80s. The Riemann hypothesis is equivalent to

$$\log g(n) < \sqrt{Li^{-1}(n)} \text{ (for sufficiently large n)}$$

where $g(n)$ is the maximum order of an element of the symmetric group S_n and Li^{-1} is the inverse function of Li. At the end of each of the chapters of the first volume are open problems.

Volume Two is devoted to analytic equivalents. As an early example, M. Riesz proved in 1916 that the Riemann Hypothesis is equivalent to the condition

$$R(x) := \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(n-1)!\zeta(2n)} x^n \ll_\epsilon x^{\frac{1}{4}+\epsilon}.$$

The topics in Volume Two contains both classical material as well as more modern developments. This volume broaches some of the more diverse range of fields as mentioned above. Chapter 12 is the sole chapter on the generalized Riemann hypothesis.

There is a suite of Mathematica software available from the author for further numerical explorations. **RHpack** is for the classical Riemann hypothesis while **GHpack** is for the generalized Riemann hypothesis.

In both volumes, some of the equivalents are to a greater or lesser degree foundational, in the sense that some more modern equivalences are consequences of older established equivalences, although their proofs can be non-trivial. Attention is given to this when appropriate.

As the MAA Reviews are especially interested in possible uses for undergraduates, it should be pointed out again that these volumes are aimed at an audience that includes graduate students and researchers and that a good graduate course in analysis would probabily be a minimal prerequisite for most readers. Nonetheless a resourceful undergraduate research mentor might be able to find some things that might be accessible to advanced undergraduates. For example, the fact that the Euler product for $\zeta(s)$ diverges when $s = 1$, entails $\sum \frac{1}{p} = \infty$ and so the infinitude of primes (this appears as an exercise in "Baby Rudin"). Use of some of the **RHpack** and **GHpack** software is something that could possibly used in projects for undergraduates.

Being a first printing, the books do have some typos/misprints, some rather glaring. The author has a list of errata on his web page. It would have been nice to have a table of symbols but all in all these two volumes are a must have for anyone interested in the Riemann Hypothesis.

Steven Deckelman is a professor of mathematics at the University of Wisconsin-Stout, where he has been since 1997. He received his Ph.D from the University of Wisconsin-Madison in 1994 for a thesis in several complex variables written under Patrick Ahern. Some of his interests include complex analysis, mathematical biology and the history of mathematics.

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Number Theory

Mahnkopf, J.

Book review of: K. Broughan, Equivalents of the Riemann hypothesis. Volume 2: Analytic equivalents. (English) Zbl 1435.00031

Monatsh. Math. 190, No. 2, 406 (2019).

Review of [Zbl 1427.11002].

MSC:

00A17 External book reviews
11-02 Research exposition (monographs, survey articles) pertaining to number theory
11M26 Nonreal zeros of $\zeta(s)$ and $L(s,\chi)$; Riemann and other hypotheses
11N05 Distribution of primes
11N25 Distribution of integers with specified multiplicative constraints

Full Text: DOI

Broughan, K.: *Equivalents of the Riemann Hypothesis* Volume Two: Analytic Equivalents (Encyclopedia of Mathematics and Its Applications 165). XX, 491 pp., Cambridge University Press, Cambridge, 2017. £120,00.

The Riemann hypothesis (RH) is among the outstanding unsolved problems in Mathematics. It is one of the seven Millenium problems and it is one of the few still unsolved problems on Hilbert's famous list. Its importance for mathematics and number theory in particular hardly can be overestimated; the answer to many questions and open problems in number theory is known to depend on whether the RH is true or not. The aim of the two volumes by Kevin Broughan is to give a selection of conjectures that are equivalent to the RH. While the first volume concentrates on equivalents which are of an arithmetical nature this second volume collects equivalents which are of an analytic origin. The result is an impressive and surprising list of conjectures about estimates on infinite series, existence of certain dense subsets of certain L^p-spaces, non-positivity of the Bruijn-Newman constant, limits of sequences of orthogonal polynomials, heights of cyclotomic polynomials, solutions of certain integral equations, and many more, which all are equivalent to the RH. Most of these equivalents can be understood on a more elementary level. In addition there is a chapter on Weil's explicit formula which leads to Weil's criterion for the truth of RH and Bombieri's variational approach to the RH. This chapter also contains a discussion of the Weil conjectures about the Zeta function of a smooth projective variety over a finite field and its solution in the case of elliptic curves. The book closes with a series of appendices explaining background results from analysis, functional analysis, measure- and integration theory and integral transforms.

This book may serve as a reference for the RH and its equivalent formulations or as an inspiration for everyone interested in number theory. It is written in a very readable style and for most parts only assumes basic knowledge from (complex) analysis. Thus, it may also serve as a (somewhat specific) introduction to analytic number theory.

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