Old and new arithmetic and analytic equivalences of the Riemann hypothesis

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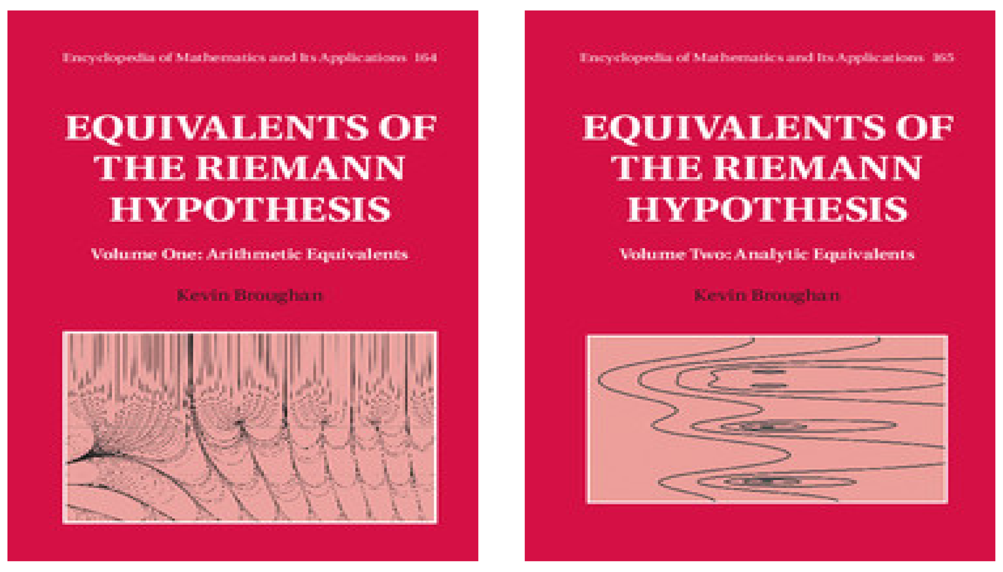
Progress on the problem during the twentieth century

- Harald Bohr and Edmund Landau in 1914 showed that the greater proportion of zeros were on or near the line.
- Hardy in 1914 proved that there were an infinite number of zeros on the line.
- Artle Selberg in 1942 developed the Hardy-Littlewood method further to show that a very small positive proportion of the zeros were on the line.
- Norman Levinson, dying of cancer in the early 1970's, proved that more than $\frac{1}{3}$ of the zeros were on the line.
- Brian Conrey, showed in 1987 that more than $\frac{2}{5}$ of the zeros were on the line.
- By 2004 it had been shown by Xavier Gordon that the first 10^{13} non-trivial zeros were on the line.

RH equivalences - why are they useful?

- They show the ubiquitous nature of RH
- The provide potential paths to resolving RH sometimes in completely different fields
- If RH is proved true, all of the equivalent statements are also true
- If RH is proved false, the negation of each of the equivalent statements is true

A study of the main equivalences of RH through 2017



(a) Vol One: Arithmetic *(b) Vol Two: Analytic*

*Figure 1 : **Equivalents of the Riemann hypothesis, Cambridge, 2017.***

Examples of arithmetic equivalences

Dixon and Spira's inequality of 1965:

If $s=\sigma+it$ **with** $\sigma>0.5$ **and** $t\geq 6.29073$ **then** $|\zeta(1-s)|>|\zeta(s)| \iff$ **RH**.

Nicolas' inequality of 1983 and 2012:

if $\beta=2+\gamma-\log\pi-2\log 2$

$$\frac{n}{\varphi(n)}<e^{\gamma}\log\log n+\frac{e^{\gamma}(1+\beta)}{\sqrt{\log n}},\quad n>2\cdots p_{120569} \iff \textbf{RH}.$$

Robin's inequality of 1984:

$$\frac{\sigma(n)}{n}<e^{\gamma}\log\log n,\quad n>5040 \iff \textbf{RH}.$$

Some other arithmetic equivalences:

- The deviation of Farey fractions from a uniform distribution (Franel/Landau)
- The average order of the sum of the Möbius $\mu(n)$ function (Littlewood)
- The determinant of the Redheffer divisibility matrix (Redheffer)
- The maximum order of the element of the symmetric group (Massias/Nicolas/Robin)

Examples of analytic equivalences

The Riesz series criterion of 1915

$$\forall\varepsilon>0,\ \textbf{as } x\to\infty,\ \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(n-1)!\zeta(2n)}x^n \ll_{\varepsilon} x^{0.25+\varepsilon} \iff \textbf{RH}.$$

Nyman-Beurling equivalence of 1955:

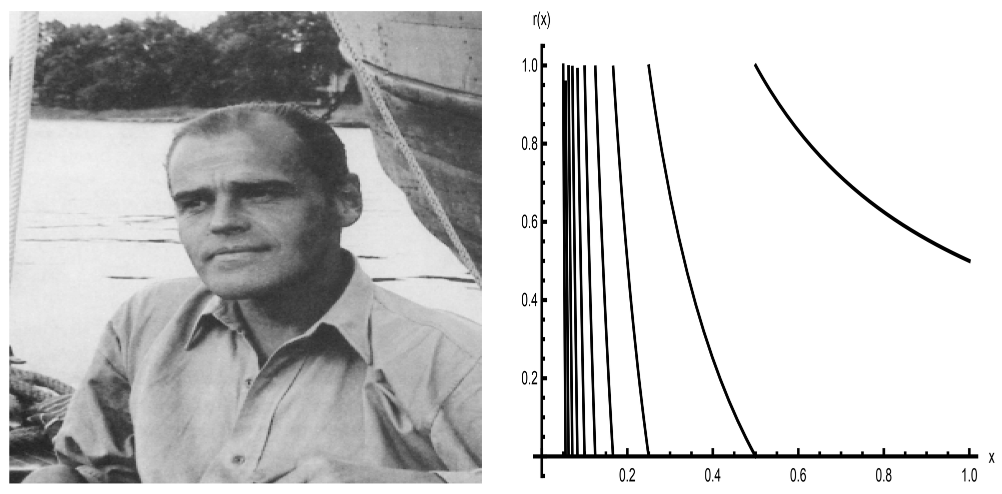
let $x>0$ and let $r(x):=\{x\}$ be the fractional part of x, so $x=\lfloor x\rfloor+r(x)$. Define a real linear space of real functions

$$\mathscr{M}:=\{f: f(x)=\sum_{n=1}^{N}a_n r\left(\frac{\theta_n}{x}\right),\ a_n\in\mathbb{R},\ \theta_n\in(0,1],\ \sum_{n=1}^{N}a_n\theta_n=0,\ N\in\mathbb{N}\}.$$

Let $1\leq p<\infty$. The subspace $\mathscr{M}$ is dense in the Banach space $L^p(0,1)$ if and only if $\zeta(s)$ has no zeros in the right half plane $\sigma>1/p$.

If $0<\sigma<1$ and $f\in\mathscr{M}$ we have

$$\zeta(s)=\frac{s}{s-1}-s\int_1^{\infty}\frac{\{x\}}{x^{s+1}}\,dx \implies \int_0^1 f(x)x^{s-1}\,dx=-\frac{\zeta(s)}{s}\sum_{n=1}^{N}a_n\theta_n^s.$$



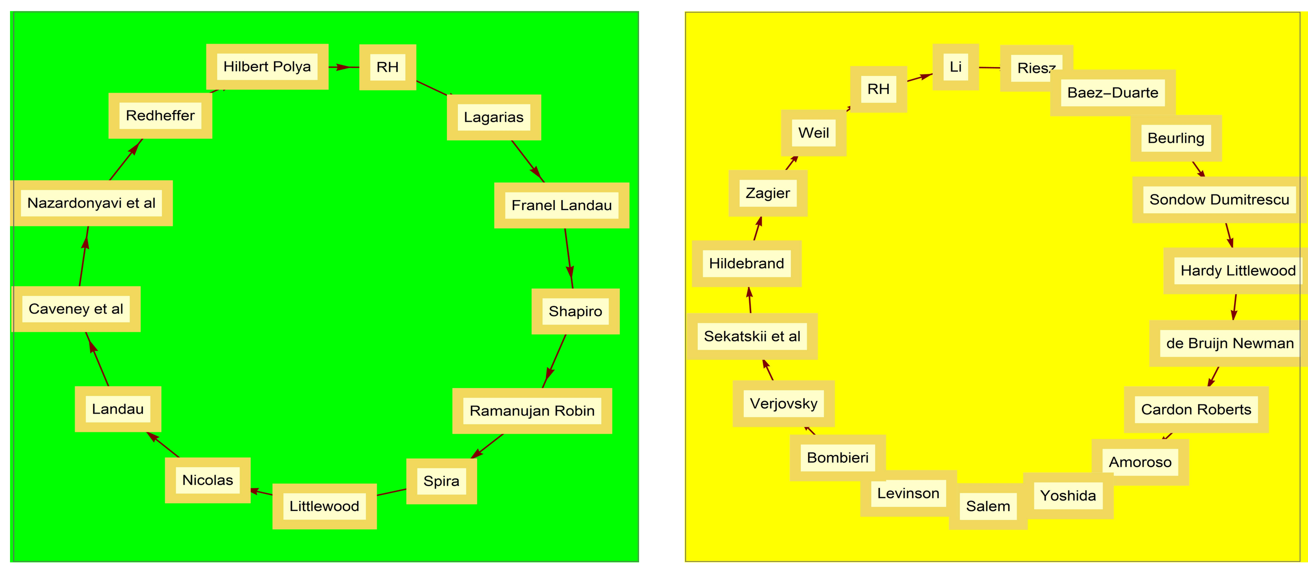
(a) Arne Beurling *(b) Values of $r(\theta/x)$ for $\theta=0.5$.*

*Figure 2 : **Beurling and the $r(\theta/x)$ function***

Some other analytic equivalences:

- $|\xi(\sigma+it)|$ is monotonic for $\sigma>0.5$ and all fixed t (Sondow/Dumitrescu).
- For all $n\in\mathbb{Z}$ we have $\sum_{\rho}\left(1-\left(1-\frac{1}{\rho}\right)^n\right)>0$ (Li, Bombieri/Lagarias).
- Integral equations (Sekatskii/Beltraminelli/Merlini, Salem, Levinson).
- Equivalences based on Weil's explicit formula (Weil).
- Discrete measure equivalences (Verjovsky).
- Smooth number equivalence (Hildebrand).
- Hermitian form equivalence (Yoshida), with insufficient room for others ....

Overview of discoverers of (most of) the main equivalences



(a) Vol One: Arithmetic Equivalents *(b) Vol Two: Analytic Equivalents*

*Figure 3 : **Named main equivalences up to 2017***

New and evolving equivalences

The new Tao/Rogers equivalence: $\Lambda=0$

The Newman-deBruijn constant constant Λ is the minimum value of λ such that if

$$\Xi(s):=\xi(0.5+is)=\int_{-\infty}^{\infty}\Phi(u)e^{ius}du \implies \int_{-\infty}^{\infty}e^{\lambda u^2}\Phi(u)e^{ius}du,$$

has only real zeros. C. M. Newman in 1976 showed $-\infty<\Lambda$ and conjectured $\Lambda\geq 0$. Then RH is equivalent to $\Lambda\leq 0$. Brad Rogers and Terence Tao have shown recently that Newman's conjecture is true, and so RH is equivalent to $\Lambda=0$, whatever definition of Λ is used. A preprint is "The De Bruijn-Newman constant is non-negative", arXiv:1801.05914v2[math.NT] (14 Feb 2018).

The recently announced result of Ono/Griffen/Rolen/Zagier

Jensen polynomials are of the form

$$J_{\gamma}^{d,n}(x):=\sum_{i=0}^{d}\binom{d}{i}\gamma_{i+n}x^i$$

where the γ_i are associated with the Taylor expansion of $\Xi(s)$, i.e. they satisfy

$$\Xi_1(x):=\frac{1}{8}\Xi\left(\frac{i\sqrt{x}}{2}\right)=\sum_{n=0}^{\infty}\frac{\gamma(n)}{n!}x^n.$$

Ken Ono, Michael Griffin, Larry Rolen and Don Zagier have shown that all but a finite number of the Jensen polynomials for the Riemann Xi function are hyperbolic, i.e. have all real zeros. A 1927 result of Polya is apparently that RH is equivalent to **all** of these polynomials being hyperbolic. This hyperbolicity has been proved for degrees $d\leq 3$. They obtained an arbitrary precision asymptotic formula for the derivatives $\Xi^{(2n)}(0)$, which allows them to prove the hyperbolicity of 100% of the Jensen polynomials of each degree. They used Hermite polynomials.

Is RH undecidable?

If RH is undecidable, then if it is and is **false** there is a zero (which we cannot find) off the critical line. This zero would provide a proof that RH is false, but there is no such proof since RH is undecidable. Therefore it is **true** but can never be proved!

Is it computable? If RH is **false** then a finite computation will find a integer which contradicts Robin's (or easier Shapiro's) inequality, so at least its semi-computable.

A likely equivalence of Bombieri

If $\vartheta(t)$ is the (continuous) argument of $\zeta(s)$ on the critical line then $Z(t)=e^{i\vartheta(t)}\zeta(0.5+it)$ is real and has the same zeros as $\zeta(s)$ on the line. If $Z(t)$ has a positive local minimum or negative local maximum then RH would be false. It is conjectured that this statement has a converse, which **if true** would provide a valuable RH equivalence.