Sheet 1 math501-17A

(1) Show that if ρ is a metric on X and $\alpha > 0$ then $\alpha \cdot \rho$ is also a metric on X.

(2) Show that in any metric space (X, ρ), for all x, y, z

$|\rho(x,y) - \rho(y,z)| \leq \rho(x,z)$.

(3) Define the absolute value $|\cdot|$ on $\mathbb{Q}$ and prove $\rho(x,y) = |x-y|$ is a metric.

(4) Use the identity $a \geq 0, b \geq 0 \Rightarrow \dfrac{a+b}{1+a+b} \leq \dfrac{a}{1+a} + \dfrac{b}{1+b}$ (prove this first) to prove $d(x,y) := \dfrac{\rho(x,y)}{1+\rho(x,y)}$ is a metric on X if ρ is a metric on X, and that it is bounded

(5) Show $\rho(x,y) := \sqrt{|x-y|}$ is a metric on $\mathbb{R}$.

(6) If p is prime and $m \in \mathbb{Z}$ let $\nu_p(m)$ be the highest power of p which divides m. Define for $\left\{ \begin{array}{l} x = \frac{m}{n} \in \mathbb{Q} \quad n \in \mathbb{Z} \\ n \neq 0 \quad\quad\quad\quad m \in \mathbb{Z} \end{array} \right.$

$$|x|_p := \frac{1}{p^{\nu_p(m) - \nu_p(n)}} \,,$$

the p-adic absolute value on $\mathbb{Q}$. Prove $|p^n|_p \to 0$ as $n \to \infty$, $|ab|_p = |a|_p |b|_p$ and $|x+y|_p \leq \max\{|x|_p, |y|_p\}$ and $|x+y|_p \leq |x|_p + |y|_p$ so $(\mathbb{Q}, |x-y|_p)$ is a metric space

(7) Show that in a Hilbert space $(H, \|\cdot\|)$

$x \perp y \Rightarrow \|x+y\|^2 = \|x\|^2 + \|y\|^2$ and

$\forall x,y \quad \|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2$.