LECTURE 20: HAHN-BANACH THEOREM

Theorem 45: [Hahn-Banach for real normed vector spaces]. Let $(M, \|\cdot\|)$ be a normed space, let $N \subset M$ be a subspace, and let

$F : N \to \mathbb{R}$ be a linear functional which is bounded on N with norm $\|F\|_N$. Then there is a bounded linear functional

$$G : M \to \mathbb{R} \quad \text{with} \quad F(f) = G(f) \quad \forall\, f \in N \quad \text{and} \quad \|G\|_M = \|F\|_N.$$

Proof: (a) First we extend F to a subspace of 1 higher dimension: let $g \in M \setminus N$, if $f', f'' \in N$

$$\begin{aligned}
& F(f' - f'') \leqslant \|F\|_N \, \|f' - f''\| \\
& \qquad\qquad = \|F\|_N \, \|(f' + g) - (f'' + g)\| \\
\Rightarrow \quad & F(f') - F(f'') \leqslant \|F\|_N \, \|f' + g\| + \|F\|_N \, \|f'' + g\| \\
\Rightarrow \quad & -\|F\|_N \, \|f'' + g\| - F(f'') \leqslant \|F\|_N \, \|f' + g\| - F(f') \\
\Rightarrow \quad & \sup\{\text{LHS} : f'' \in N\} \leqslant \inf\{\text{RHS} : f' \in N\}
\end{aligned}$$

let $\gamma \in \mathbb{R}$ satisfy $\sup\{\text{LHS}\} \leqslant \gamma \leqslant \inf\{\text{RHS}\}$ and let $f'' = f' = f$.

Then $-\|F\|_N \, \|f + g\| \leqslant F(f) + 1.\gamma \leqslant \|F\|_N \, \|f + g\|$ — (1)

(b) Let $V = \{f + \alpha g : f \in N, \alpha \in \mathbb{R}\}$. Then N is a subspace of V which is also a subspace of M. If $x \in V$ then

$$x = f + \alpha g = f' + \alpha' g \Rightarrow N \ni f - f' = g(\alpha' - \alpha) \quad .$$

If $\alpha \neq \alpha'$ then $g \in N$ which is false. Thus $\alpha = \alpha'$ and also $f = f'$ after cancelling $\alpha g = \alpha' g$.

Therefore given $x \in V$, f and α are well determined.

Define $G : V \to \mathbb{R}$ by $G(x) = F(f) + \alpha.\gamma$. Then G is linear and if $x \in N$, $\alpha = 0$ and so $G(x) = F(x)$. We need to prove that $\|G\|_V = \|F\|_N$.

Firstly if $\alpha = 1$ by equation (1) above

$$|F(f) + 1.\gamma| \leq \|F\|_N \, \|f + 1.g\|$$

$$\|$$

Hence $\quad |G(x)| \leq \|F\|_N \, \|x\| \quad - (2) \quad$ since $x = f + 1.g$.

Therefore $\|G\|_V \leq \|F\|_N$. The opposite inequality is almost immediate since G is an extension of F. Next replace f by $-f$ in (1) to obtain (2) for $x = f - g$.

Case III: $\quad \alpha > 0$ replace f by f/α in (1)

Case IV: $\quad \alpha < 0$ replace f by $f/-\alpha$ in (1).

(c) The next step is radically different from the previous two and involves a so called transfinite construction.

Let $(P,\leq)$ be a <u>partially ordered set</u>

i.e. $\quad x \leq x$, $x \leq y$ and $y \leq x \Rightarrow x = y$, and

$x \leq y$, $y \leq z \Rightarrow x \leq z$ for all $x,y,z \in P$.

We say $a \in P$ is <u>maximal</u> if $\forall b$, $a \leq b \Rightarrow a = b$.

A subset $A \subset P$ is a <u>chain</u> or linearly ordered subset if $\forall a, b \in A$, $b \leq a$ or $a \leq b$. An element $b \in P$ is an <u>upperbound</u> for a subset $A \subset P$ if $b \geq a \quad \forall a \in A$.

We require the following set theoretic axiom: Zorn's "Lemma": If each chain in P has an upperbound in P then P has a maximal element.

Let $P = \{(V,H) : N \subset V \subset M$, V a subspace, $H : V \to \mathbb{R}$ bounded and linear extending F on N and $\|H\|_V = \|F\|_N\}$.

Define $(V,H) < (V',H')$ if $V \subset V'$ and $H'(x) = H(x) \;\; \forall x \in V$ and $\|H'\|_{V'} = \|H\|_V$.

Since $(N,F) \in P$, $P \neq \emptyset$.

It is not difficult to verify that $<$ is indeed a partial order on P.

Firstly we show that each chain in P has an upper bound. Let $\{(V_\lambda, H_\lambda)\}$ be a chain indexed by $\lambda \in \Lambda$. Let $V = \cup\, \{V_\lambda : \lambda \in \Lambda\}$.

Then V is a subspace: if $x, y \in V$ then $\exists\, \lambda$ with $x \in V_\lambda$ and a γ with $y \in V_\gamma$. Then $V_\lambda \subset V_\gamma$ or $V_\gamma \subset V_\lambda$; in either case $x + y \in V_\lambda \cup V_\gamma \subset V$. Define G on V as follows: if $x \in V$ then $x \in V_\lambda$ some λ. Let $G(x) = H_\lambda(x)$. Then G is well defined since if $x \in V_\gamma$ also we must have $H_\lambda(x) = H_\gamma(x)$.

Then (V,G) is an upper bound for the chain since given $\lambda \in \Lambda$, (1) $V_\lambda \subset V$ and $\forall\, x \in V_\lambda$, (2) $G(x) = H_\lambda(x)$, and lastly (3) $\|G\|_V = \|H_\lambda\|_{V_\lambda}$: to see this let $x \in V$. Then $x \in V_\beta$ for some β.
Thus $|G(x)| = |H_\beta(x)| \leq \|H_\beta\|_{V_\beta} \|x\| = \|H_\lambda\|_{V_\lambda} \|x\|$, since given any pair β, λ we must have $(V_\lambda, H_\lambda) \leq (V_\beta, H_\beta)$ or the reverse and so $\|H_\beta\|_{V_\beta} = \|H_\lambda\|_{V_\lambda}$.
Therefore $\|G\|_V \leq \|H_\lambda\|_{V_\lambda}$. The reverse inequality follows as before.
By (1), (2), (3) (V,G) is an upper bound for the chain.

(d) By Zorn there is a maximal element (W,H) say in P.

If $W \neq M$ we can extend by 1 dimension as in step (a) above obtaining

$$W \subset W' \subset M \quad \text{with} \quad W \neq W'$$

and then extend H to H' so that $\|H\|_W = \|H'\|_{W'}$ as above. But then $(W,H) \underset{\neq}{<} (W',H')$ which is impossible since (W,H) is maximal. Therefore $W = M$ and H extends F to M. This completes the proof.

Example: given $x \neq 0$ in $\mathbb{R}^n$ it is easy to find a (continuous) linear map π with $\pi(x) \neq 0$. However for infinite dimensional normed spaces we need the power of Hahn-Banach to show that such a continuous π exists.