Burnside-Polya counting example

In how many <u>non equivalent</u> ways can you design p beads on a necklass with $a \geq 1$ colours?

$$\mathrm{fix}(g) = \{ u \in U : g(u) = u \}.$$

$G = \mathbb{Z}_5$, U = all colourings for 2 colours.

$$\# \text{orbits} = \sum_{i \in G} \frac{|\mathrm{fix}(i)|}{|G|} \qquad |G| = 5.$$

$\mathrm{fix}(i) = \{$  $\}$. $\quad 1 \leq i \leq p-1$

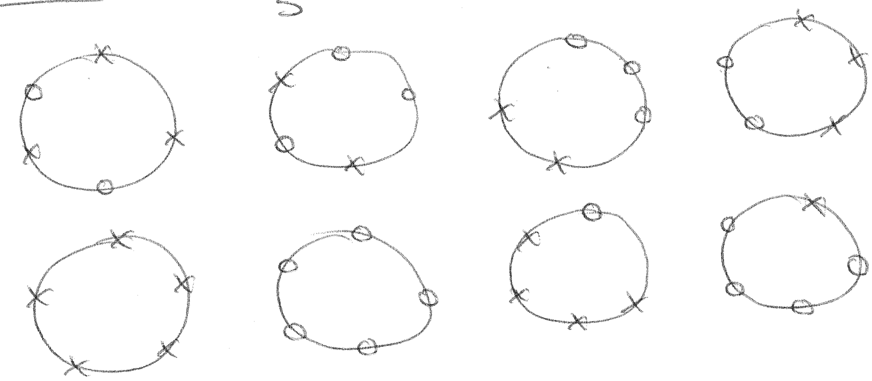
$$\mathrm{fix}(0) = \{\text{All coloured beads}\} = 2^5.$$

$$\therefore \quad \# \text{orbits} = \frac{(5-1)2 + 2^5}{5}$$

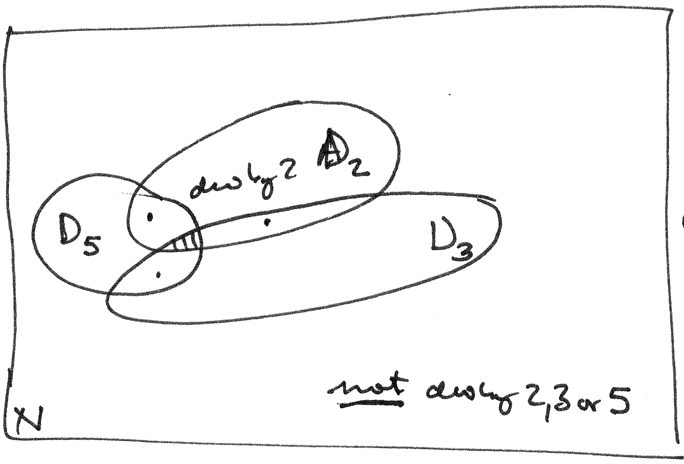
In general for $a = 1, 2, \ldots$ colours and $p \in \mathbb{P}$ beads.

$$\boxed{\# \text{orbits} = \frac{(p-1)a + a^p}{p}}$$

Check: $\dfrac{(5-1)2 + 2^5}{5} = 2 + \dfrac{2^5 - 2}{5} = 8$



Ex.



Inclusion / Exclusion Example

Count the numbers up to 20 not divisible by 2, 3 or 5
e.g. 1, 7, … ?

$$\# \text{ not div} = N - (|D_2| + |D_3| + |D_5|) + (|D_2 \cap D_3| + |D_2 \cap D_5| + |D_3 \cap D_5|) - |D_2 \cap D_3 \cap D_5|$$

$N = 20$.

$D_2 = \{2, 4, 6, 8, 10, 12, 14, 16, 18, 20\}$ $\qquad \# = 10 \; \left\lfloor \frac{20}{2} \right\rfloor$

$D_3 = \{3, 6, 9, 12, 15, 18\}$ $\qquad \# = 6 = \left\lfloor \frac{20}{3} \right\rfloor$

$D_5 = \{5, 10, 15, 20\}$ $\qquad \# = 4$

$D_2 \cap D_3 = \{6, 12, 18\}$ $\qquad \# = 3$

$D_2 \cap D_5 = \{10, 20\}$ $\qquad \# = 2$

$D_3 \cap D_5 = \{15\}$ $\qquad \# = 1$

$D_2 \cap D_3 \cap D_5 = \emptyset.$ $\qquad \# = 0.$

$\therefore \quad \# \text{ not div} = 20 - (10 + 6 + 4) + (3 + 2 + 1) - 0$

$= 6$

set of non div $= \{1 \; \cancel{2} \; \cancel{3} \; \cancel{4} \; \cancel{5} \; \cancel{6} \; 7 \; \cancel{8} \; \cancel{9} \; \cancel{10} \; 11 \; \cancel{12} \; 13 \; \cancel{14} \; \cancel{15} \; \cancel{16} \; 17 \; \cancel{18} \; 19 \; \cancel{20}\}$

$= \{1, 7, 11, 13, 17, 19\}$.