Generating Functions

web : http://www.math.waikato.ac.nz/~kab

§1. Part of enumeration = counting $\subset$ combinatorics

classical enumerative combinatorics $\rightarrow$ Gian-Carlo-Rota, Richard Stanley – $\begin{cases} \text{conceptual} \\ \text{structural} \\ \text{algebraic} \end{cases}$

modern $\begin{cases} \text{explicit} \leftarrow \text{computer algorithms} \\ \text{concrete} \\ \text{constructive} \end{cases}$

Def^n If A is a set then $|A|$ is the number of elements in A
so $|A| \in \{0,1,2,\dots\} = \mathbb{Z}^+ = \mathbb{N} \cup \{0\} = \mathbb{N}'$

Funamental Theorem $\quad |A| = \sum_{a \in A} 1 .$

But, if a set is a family of elements with one property, we want to count sets $(A_n : n \in \mathbb{N}')$ with a property depending on n.

Fundamental Question How many elements does A_n have ?

But what is an answer ?

Examples Def^n $\quad a_n := |A_n|$

(1) I. Ching : let A_n be the set of subsets of $\{1, 2, \dots, n\}$

Ex $\quad n = 3 \quad A_3 = \{ \emptyset,$
$\{1\}, \{2\}, \{3\},$
$\{1\ 2\}, \{1\ 3\}, \{2, 3\},$
$\{1, 2, 3\} \}$

$|A_3| = 8 = 2^3,$

$a_n = 2^n .$

(2) Rabbi Levi Ben Gerson A_n is the set of all permutations of $\{1,2,3,\ldots n\}$.

Ex (1 2 3) (1 3 2) , $a_3 = 6 = 3!$, $a_n = n!$
(2 1 3) (2 3 1)
(3 1 2) (3 2 1)

This is the size of the symmetric group S_n.

(3) Catalan (of Catalan's Conjecture fame : $3^2 - 2^3 = 1$, the only consecutive powers)

A_n is the set of well formed bracketings with n left and n right brackets.

Ex $n = 3$ ( )( )( ) $a_3 = 5$, $a_n = \dfrac{(2n)!}{(n+1)!\ n!}$
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(4) (Leonardo of Pisà) Fibonacci Let A_n be the set of finite sequences of only 1s or 2s that sum to n.

Ex $n = 4$ 1111 $a_4 = 5$, $a_n = F_{n+1}$ the (1+n)th Fibonacci number
112
121
211
22

$F_0 = 0,\ F_1 = 1,\ F_n = F_{n-1} + F_{n-2}$

$\{F_n : n \in \mathbb{N}'\} = (0, 1, 1, 2, 3, 5, \ldots)$

Also $a_n = \displaystyle\sum_{j=0}^{\lfloor n/2 \rfloor} \binom{n-j}{j}$ $\lfloor x \rfloor = \max\{m \in \mathbb{Z} : m \leq x\}$ = integer part of $x \in \mathbb{R}$

Also $a_n = \dfrac{1}{\sqrt{5}}\left(\left(\dfrac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\dfrac{1-\sqrt{5}}{2}\right)^{n+1}\right)$

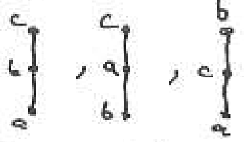
E.g. $a_1 = \dfrac{1}{\sqrt{5}}\dfrac{1}{2^2}\left[(1+\sqrt{5})^2 - (1-\sqrt{5})^2\right]$

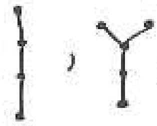
$= \dfrac{1}{4\sqrt{5}}\left[(1+\sqrt{5}+1-\sqrt{5})(1+\sqrt{5}-1+\sqrt{5})\right]$

$= 1 = F_2$.

(5) <u>Cayley</u> A_n is the set of <u>labelled trees</u> on n vertices. A "tree" is a connected graph with no cycles. It is "labelled" if each vertex has a distinct name

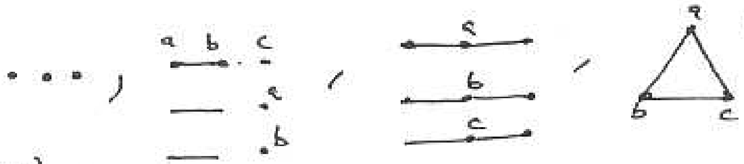
<u>Ex.</u> $a_2 = |A_2| = | \{$ $\}| = 1 = 2^{2-2}$

$a_3 = |A_3| = | \{$  $\}| = 3^{3-2} = 3$

$a_4 = | \{$  $, \dots \}| = 4^{4-2} = 16$

$$a_n = n^{n-2}$$

(6) A_n is the set of labelled <u>simple</u> graphs with n vertices i.e. graphs which have no <u>loops</u> or <u>multiple edges</u>.

$A_3 = \{$  $\} \Rightarrow a_3 = 8 = 2^3$

$$a_n = 2^{\frac{n(n-1)}{2}}$$

(7) A_n is the set of labelled <u>connected</u> simple graphs on n vertices i.e. every pair of vertices is connected by a path.

$A_3 = \{$  $\}$, $a_3 = 4$

then $a_n = n! \times$ coefficient of x^n in the series expansion of

$$\log\left(\sum_{j=0}^{\infty} \frac{2^{j(j-1)/2}}{j!} x^j \right) = x + \frac{x^2}{2} + \frac{2x^3}{3} + \frac{19x^4}{12} + O(x^5)$$

so $a_3 = 3! \times \frac{2}{3} = 4$.

(8) A_n is the number of <u>Latin squares</u> of size n i.e. each row and each col contains contains each of $\{1, \dots, n\}$ once and only once.

$$a_1 = 1, \quad a_2 = \left| \left\{ \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 2 & 1 \\ \hline \end{array}, \begin{array}{|c|c|} \hline 2 & 1 \\ \hline 1 & 2 \\ \hline \end{array} \right\} \right| = 2, \quad a_n = ?$$

So the question is "what is a_n?"

The answer according to Wilf [1982]: an algorithm, polynomial or better in n, to compute (a_n).

We always have the formula: $a_n = \sum_{a \in A_n} 1$

(Brute force, greedy, caveman's method)

So what is a formula?

Ex (1) $a_n = 2^n$

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input: n
  if n = 0   a_n = 1,
     else    a_n = 2 a_{n-1}
output a_n
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number of steps $= n$ i.e. an $O(n)$ algorithm if $\times 2$ takes one unit of time and space is unbounded.

(but what if $n = 10^{10^{10}}$!)

But

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if n = 0          a_n = 1
else if n is odd       a_n = 2 a_{n-1}
else     a_n = a_{n/2}^2
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an $O(\log n)$ algorithm. Ex $n = 10^{10} \to 10$ steps vs 10^{10}.

Applications of Enumeration

- Probability and statistics: prob $\{ i \in A_n \}$ is $\frac{a_n}{n}$
- number theory $A_n = \{ 1 \le j \le n : (j, n) = 1 \}$, $a_n = \varphi(n)$.
- computational complexity of algorithms
  i.e. the number of steps needed to execute an algorithm.

7 Enumeration Methods

Disjoint union: If $A \cap B = \emptyset$ write $A \sqcup B := A \cup B$

Ex $\{1,2,3\} \cup \{3,4,5\} = \{1,2,3,4,5\}$

but $\{1,2,3\} \sqcup \{3,4,5\}$ is not defined.

Decomposition rule $|A \sqcup B| = |A| + |B|$

$$\left| (\cdot\cdot\cdot) \; (\cdot\cdot) \right| = 5$$

Product rule $|A \times B| = |A| \cdot |B|$

Ex $A = \{a, b, c\}$, $B = \{1, 2\}$

$$A \times B = \{(a,1), (a,2), (b,1), (b,2), (c,1), (c,2)\}$$

$$|A \times B| = 6 = 3 \times 2 = |A| \cdot |B|.$$

Function Rule let $A^B = \{ f : f : B \to A \text{ is a function} \}$

Then for each $a \in B$ we can choose $f(a)$ in $|A|$ ways. Hence

$$|A^B| = \underbrace{|A| \cdot |A| \cdot \cdots |A|}_{|B| \text{ copies}} = |A|^{|B|}$$

Ex A_n = set of sequences of 0's and 1's of size n

$n = 3$

$$A_3 = \left\{ \begin{matrix} 000, & 110, & 111 \\ 100, & 101, & \\ 010, & 011, & \\ 001, & & \end{matrix} \right\} \Rightarrow a_3 = 2^3 = 8$$

$B = \{a, b, c\}$, $A = \{0, 1\}$

$$f : B \to A \leftrightarrow (f(a)\; f(b)\; f(c))$$

Hence $a_n = |A|^{|B|} = 2^n = |\mathcal{P}(\{1, 2, \ldots n\})|$ (1)

power set

since

$$\{b_1, \ldots, b_n\} \longleftrightarrow b_i \in S \subset \{1, 2, \ldots, n\}$$

$$e_1 e_2 \ldots e_n \qquad \Leftrightarrow e_i = 1$$

Refinement (Glorified disjoint union)

If $A_n = \bigsqcup_{j \in J} B_{n,j}$

and $b_{n,j} = |B_{n,j}| \Rightarrow a_n = \sum_{j \in J} b_{n,j}$

Ex (4) (Sequences of 1's & 2's summing to n)

Let $B_{n,j} = \{$set of all sequences in A_n with exactly $j \geq 0$, 2's$\}$

If $\exists$ j 2's then there are $n-2j$ 1's so

$b_{n,j}$ = # ways of selecting the j 2's from $\left.n-2j+j = n-j\right\}$

$$= \binom{n-j}{j}$$

Hence $a_n = \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-j}{j}$

Recursion If A_n is a 'combination' of operations on $A_0, A_1, \ldots A_{n-1}$

we may get an equation, a so called recurrence relation,

$$a_n = P(a_{n-1}, a_{n-2}, \ldots a_0)$$

Ex(4) A sequence can start with 1 or 2 but not both.
If it starts with 1, the rest adds up to $n-1$ so any seq. in A_{n-1} will do.
If it starts with 2, the rest adds up to $n-2$, i.e. A_{n-2} seq.

$\therefore$ $A_n = 1A_{n-1} \sqcup 2A_{n-2}$ and $|1A_{n-1}| = |A_{n-1}| = a_{n-1}$
$|2A_{n-2}| = |A_{n-2}| = a_{n-2}$

$\therefore$ $a_n = a_{n-1} + a_{n-2}$

Finally $a_0 = |A_0| = |\emptyset| = 0$
$a_1 = |A_1| = |\{1\}| = 1$