The University of Waikato
Department of Mathematics

Topics in Pure Mathematics: Generating Functions, math319-10B Assignment 1, due Friday 6th August 2010 - through the 252 slot outside G3.19.

1. Find the ordinary power series generating function for each of the following sequences, in simple, closed form as a function of x. In each case the sequence is defined for all integers $n \geq 0$:

$$\begin{aligned} &(a) \quad a_n = 5n + 7, \\ &(b) \quad a_n = n^2. \end{aligned}$$

2. Find the ordinary power series generating function for each of the following sequences given by a recurrence relation, in simple, closed form and hence solve the recurrence by finding a formula for a_n as a function of n:

$$\begin{aligned} &(a) \quad a_{n+1} = 3a_n + 2, \ a_0 = 0, \\ &(b) \quad a_{n+2} = 2a_{n+1} - a_n, \ a_0 = 0, \ a_1 = 1, \\ &(c) \quad a_{n+1} = 3a_n + 2n, \ a_0 = 1. \end{aligned}$$

1 (a)

$a_n = 5n+7 \Rightarrow$

$$f(x) = \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} (5n+7)x^n = 5\sum_{n=0}^{\infty} nx^n + 7\sum_{n=0}^{\infty} x^n = \boxtimes$$

But $\frac{1}{1-x} = 1 + x + x^2 + \dots + x^n + \dots$ for $x \in \mathbb{R}$, $|x|<1$ and

$$= \sum_{n=0}^{\infty} x^n$$

$$\left(\frac{1}{1-x}\right)^2 = 1 + 2x + 3x^2 + \dots + (n+1)x^n + \dots \qquad |x|<1$$

$$\therefore \frac{x}{(1-x)^2} = x + 2x^2 + 3x^3 + \dots + nx^n + \dots \quad \circledast$$

$$= \sum_{n=1}^{\infty} nx^n = \sum_{n=0}^{\infty} nx^n$$

Hence $f(x) = \frac{5x}{(1-x)^2} + \frac{7}{1-x} = \frac{5x + 7(1-x)}{(1-x)^2} = \frac{7-2x}{(1-x)^2}$ //

(b) $a_n = n^2$. By $\circledast$ $\left(\frac{x}{(1-x)^2}\right)' = 1 + 2^2x + 3^2x^2 + 4^2x^3 + \dots n^2x^{n-1} + \dots$

$$\Rightarrow \quad x\left(\frac{x}{(1-x)^2}\right)' = 1^2x + 2^2x^2 + 3^2x^3 + 4^2x^4 + \dots n^2x^n + \dots$$

$$\therefore f(x) = \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} n^2x^n = \sum_{n=1}^{\infty} n^2x^n = x\left(\frac{x}{(1-x)^2}\right)'$$

$$= \frac{x(1+x)}{(1-x)^3}$$

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2 $a_{n+1} = 3a_n + 2$, $a_0 = 0$

$$f(x) = \sum_{n=0}^{\infty} a_n x^n = \sum_{n=1}^{\infty} a_n x^n = x\sum_{n=1}^{\infty} a_n x^{n-1} = x\sum_{m=0}^{\infty} a_{m+1} x^m$$ on letg $m=n-1$

$$= x\sum_{m=0}^{\infty}(3a_m+2)x^m = 3x\sum_{m=0}^{\infty} a_m x^m + 2x\sum_{m=0}^{\infty} x^m$$

$$= 3x\,f(x) + \frac{2x}{1-x} \Rightarrow f(x)(1-3x) = \frac{2x}{1-x}$$

$$\Rightarrow f(x) = \frac{2x}{(1-x)(1-3x)}.$$

Let $\dfrac{2x}{(1-x)(1-3x)} = \dfrac{A}{1-x} + \dfrac{B}{1-3x} \Rightarrow 2x = A(1-3x) + B(1-x)$

$$\begin{cases} x = 1 \Rightarrow A = -1 \\ x = \frac{1}{3} \Rightarrow B = 1 \end{cases}$$

Hence $f(x) = \dfrac{1}{1-3x} - \dfrac{1}{1-x}$

$$= \sum_{n=0}^{\infty}(3x)^n - \sum_{n=0}^{\infty} x^n = \sum_{n=0}^{\infty}(3^n-1)x^n = \sum_{n=0}^{\infty} a_n x^n$$

Hence $a_n = 3^n - 1$. Check $a_0 = 3^0 - 1 = 1 - 1 = 0$

& $a_1 = 3^1 - 1 = 2$

Is $a_1 = 3a_0 + 2$?

$2 = 3.0 + 2$ ✓

(b) $a_{n+2} = 2a_{n+1} - a_n$, $a_0 = 0$, $a_1 = 1$.

$$f(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + \sum_{n=2}^{\infty} a_n x^n = x + x^2\sum_{m=0}^{\infty} a_{m+2} x^m$$

$$= x + x^2\sum_{m=0}^{\infty}(2a_{m+1} - a_m)x^m$$

$$= x + 2x^2\sum_{m=0}^{\infty} a_{m+1}x^m - x^2\sum_{m=0}^{\infty} a_m x^m$$

$$= x + 2x\sum_{m=0}^{\infty} a_{m+1}x^{m+1} - x^2 f(x).$$

$$= x + 2x(f(x) - \underset{0}{\underset{\shortparallel}{a_0}}) - x^2 f(x)$$

$$\Longrightarrow f(x)[1 - 2x + x^2] = x \Rightarrow f(x) = \frac{x}{(1-x)^2}$$

$$\left(\frac{1}{1-x}\right)^2 = 1 + 2x + 3x^2 + \cdots (n+1)x^n + \cdots \Rightarrow \frac{x}{(1-x)^2} = x + 2x^2 + 3x^3 + \cdots + nx^n + \cdots$$

Hence $a_n = n \quad \forall n \geq 0$

Check $(n+2) = 2(n+1) - n, \forall n \geq 0.$

2(c) $\quad a_{n+1} = 3a_n + 2n \;, \quad a_0 = 1$

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n x^n = 1 + x\sum_{n=1}^{\infty} a_n x^{n-1} = 1 + x\sum_{m=0}^{\infty} a_{m+1} x^m$$

$$= 1 + x\sum_{m=0}^{\infty} (3a_m + 2m)x^m$$

$$= 1 + 3x\sum_{m=0}^{\infty} a_m x^m + 2x\sum_{m=0}^{\infty} m x^m$$

$$= 1 + 3x f(x) + 2x\frac{x}{(1-x)^2} \quad (\leftarrow \text{see back})$$

$$\Rightarrow f(x)[1-3x] = 1 + \frac{2x^2}{(1-x)^2}$$

$$\Rightarrow f(x) = \frac{1}{1-3x} + \frac{2x^2}{(1-x)^2(1-3x)}$$

$$= \frac{(1/2)}{1-x} - \frac{1}{(1-x)^2} + \frac{(3/2)}{1-3x}$$

$$= \frac{1}{2}\sum_{n=0}^{\infty} x^n - \sum_{n=0}^{\infty}(n+1)x^n + \frac{3}{2}\sum_{n=0}^{\infty} 3^n x^n$$

$$= \sum_{n=0}^{\infty}\left[\frac{1}{2} - (n+1) + \frac{3}{2}3^n\right]x^n$$

$$= \sum_{n=0}^{\infty} \frac{1}{2}\left[3^{n+1} - 2(n+1) + 1\right]x^n.$$

Hence $\quad a_n = \frac{1}{2}(3^{n+1} - 2(n+1) + 1)$

Check: $\quad a_0 = \frac{1}{2}(3^1 - 2.1 + 1) = 1.$ //