1. $f(x) = x^3 + x^2 - 12$ (*)

$d \mid 12 \Rightarrow d = \pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 12$

& the only d giving $f(d) = 0$ is $d = 2$

Hence $x - 2 \mid f(x)$

$$
\begin{array}{r l}
 & x^2 + 3x + 6 \\
x - 2 \,) & x^3 + x^2 + x - 12 \\
 & x^3 - 2x^2 \\
\hline
 & 3x^2 \\
 & 3x^2 - 6x \\
\hline
 & 6x - 12 \\
 & 6x - 12 \\
\hline
 & \cdot \quad \cdot
\end{array}
$$

Thus $f(x) = (x-2)(x^2 + 3x + 6)$. By Eisenstein using $p = 3$, $x^2 + 3x + 6$ is irreducible over $\mathbb{Q}$.

But $(x^2 + 3x + 6) = (x - \alpha)(x - \beta)$ $\alpha, \beta \in \mathbb{C}$

& $\alpha, \beta = \dfrac{-3 \pm \sqrt{9 - 24}}{2} = \dfrac{-3 \pm i\sqrt{15}}{2}$.

Vieta's Method: First get rid of the x^2 term by subst. $y = x + \frac{1}{3}$

& then (*) $\Rightarrow$ $g(y) = y^3 \overset{p}{-\frac{1}{3}} y \overset{q}{-\frac{322}{27}} = 0$

so we have to solve $3ab = -\frac{1}{3} \Rightarrow ab = -\frac{1}{9}$

& $a - b = y$

Working back: $x = 2 \Rightarrow y = 7/3$ is a sol^n so solve

$ab = -\frac{1}{9}$ $a - b = \frac{7}{3}$ for a & b & get

$a = \dfrac{7 - 3\sqrt{5}}{6} \Rightarrow a^3 = \dfrac{161}{27} - \dfrac{8\sqrt{5}}{3} = \theta$.

$b = \dfrac{-7 - 3\sqrt{5}}{6}$

& $27\theta^2 + 27\theta q - p^3 = \dots = 0$

so its all consistent (but I could not derive a root directly!).

(2) $\alpha = \sqrt{3} + \sqrt{-5}$

$\Rightarrow \alpha^2 = 3 + (-5) + 2\sqrt{-15} \Rightarrow \alpha^2 + 2 = 2\sqrt{-15}$

$\Rightarrow (\alpha^2+2)^2 = 4(-15) = -60$

$\Rightarrow \alpha^4 + 4\alpha^2 + 4 + 60 = 0$

$\alpha^4 + 4\alpha^2 + 64 = 0$

Claim: this is irreducible. If it had a root then it would have a root mod 5 but

$\alpha^4 + 4\alpha^2 + 64 \equiv \alpha^4 + 4\alpha^2 + 4 =$

	α
4	1
1	2
1	3
4	4
4	5

$\neq 0$.

Hence any factor is of degree 2 in α

$x^4 + 4x^2 + 64 = (x^2 + ax + b)(x^2 + cx + d)$

equating coeff: (1) $a + c = 0 \Rightarrow c = -a$

(2) $bc + ad = 0 \Rightarrow b = d$

(3) $b + d + ac = 4 \Rightarrow 2b = a^2 \Rightarrow b > 0$

(4) $bd = 64 \Rightarrow b = d = 8 \Rightarrow a = \pm 4 \Rightarrow c = \mp 4$

But (3) is then $16 - 16 = 0 = 4$ claim is false

$\therefore f(x) = x^4 + 4x^2 + 64$ is irreducible & since it is monic, is the minimal poly of α. //

(3) $\left.\begin{array}{l} \sqrt{2} \in \mathbb{Q}(\sqrt{2},\sqrt{3}) \\ \sqrt{3} \in \mathbb{Q}(\sqrt{2},\sqrt{3}) \\ \mathbb{Q} \subset \mathbb{Q}(\sqrt{2},\sqrt{3}) \end{array}\right\} \Rightarrow \overbrace{\mathbb{Q}(\sqrt{2}-\sqrt{3})}^{\mathbb{K} :=} \subset \mathbb{Q}(\sqrt{2},\sqrt{3})$ since $\mathbb{Q}(\sqrt{2}-\sqrt{3})$ is the smallest field containing $\mathbb{Q}$ & $\sqrt{2}-\sqrt{3}$.

now $(\sqrt{3}-\sqrt{2})(\sqrt{3}+\sqrt{2}) = 3 - 2 = 1 \Rightarrow \sqrt{2}+\sqrt{3} = \dfrac{1}{\sqrt{3}-\sqrt{2}} \in \mathbb{Q}(\sqrt{2}-\sqrt{3}) = \mathbb{K}$ since it is a field.

Hence $(\sqrt{2}+\sqrt{3}) + (\sqrt{2}-\sqrt{3}) = 2\sqrt{2} \in \mathbb{K} \Rightarrow \sqrt{2} \in \mathbb{K}$

& $(\sqrt{2}+\sqrt{3}) - (\sqrt{2}-\sqrt{3}) = 2\sqrt{3} \in \mathbb{K} \Rightarrow \sqrt{3} \in \mathbb{K} \Rightarrow \mathbb{Q}(\sqrt{2},\sqrt{3}) \subset \mathbb{K} \Rightarrow$

$\mathbb{Q}(\sqrt{2},\sqrt{3}) = \mathbb{K}$ //

(PS3)

(4) $f(x)=x^3-2$. Let $\omega = e^{\frac{2\pi i}{3}} = \frac{-1+i\sqrt{3}}{2}$ be the principal cube root of 1.

Then $\theta = 2^{1/3}\omega$ satisfies $f(\theta) = (2^{1/3}\omega)^3 - 2 = (2^{\frac{1}{3}})^3\omega^3 - 2$
$= 2.1 - 2 = 0$

So is a root of $f(x)$ in $\mathbb{C}$.

and, by Eisenstein with $p=2$, $f(x)$ is irreducible over $\mathbb{Q}$

and its monic. We can factor $f(x)$ as.

$$f(x) = (x-2^{1/3})(x-2^{1/3}\omega)(x-2^{\frac{1}{3}}\omega^2)$$

Since

$$\begin{cases} f(2^{1/3}) = (2^{1/3})^3 - 2 = 0 \\ f(2^{1/3}\omega) = (2^{1/3}\omega)^3 - 2 = 2\omega^3 - 2 = 2.1-2 = 0 \\ f(2^{1/3}\omega^2) = 0 \end{cases}$$

and each of the complex numbers is distinct. $\{2^{1/3}, 2^{1/3}\omega, 2^{1/3}\omega^2\}$

Hence $f(x)$ splits in $\mathbb{Q}(2^{1/3},\omega) := \mathbb{K}$

We can use the primitive element thm to show $\mathbb{K} = \mathbb{Q}(2^{1/3}+\omega)$

(5) Let $\alpha = \sqrt{1+\sqrt{3}} \Rightarrow \alpha^2 - 1 = \sqrt{3} \Rightarrow (\alpha^2-1)^2 = 3$

$\Rightarrow \alpha^4 - 2\alpha^2 + 1 = 9$ so. $\alpha^4 - 2\alpha^2 - 8 = 0$

Let. $f(x) = x^4 - 2x^2 - 8$. A root r divides 8 so $r \in \{\pm1, \pm2, \pm4, \pm8\}$

and $f(r) \neq 0$ for each of these possible values. The only ~~...~~ factor is quadratic. A similar proof to (2) above shows this is impossible

$$\left.\begin{aligned} a+c &= 0 \\ ad+bc &= 0 \\ b+d+ac &= -2 \\ bd &= -8 \end{aligned}\right\} \Rightarrow b^2 = -8 \;(!!).$$

$$\therefore \mathbb{Q}(\sqrt{1+\sqrt{3}}) \cong \frac{\mathbb{Q}[x]}{\langle x^4-2x^2-8 \rangle}$$

6 Let $f(x) = x^3 - 2$ so, by Eisenstein, $f(x)$ is irreducible over $\mathbb{Q}$. $f(2^{1/3}) = 0$

& $\mathbb{K} := \mathbb{Q}(2^{1/3}) \cong \dfrac{\mathbb{Q}[x]}{\langle x^3-2 \rangle}$ is generated by $\{1, 2^{1/3}, 2^{2/3}\}$ as a basis. $\boxtimes$

$\therefore \exists a,b,c \in \mathbb{Q}$ so $\dfrac{1+2^{2/3}}{2-2^{1/3}} = a + b2^{1/3} + c2^{2/3}$

Since the LHS is an element of $\mathbb{K}$.

$\therefore \quad 1+2^{2/3} = (2-2^{1/3})(a+b2^{1/3}+c2^{2/3})$

$$1 + 0 \cdot 2^{1/3} + 1 \cdot 2^{2/3} = \begin{matrix} 2a & + 2b2^{1/3} & + 2c2^{2/3} \\ -2c & - a2^{1/3} & - b2^{2/3} \end{matrix}$$

using $\boxtimes \Rightarrow$
$$\left.\begin{aligned} 1 &= 2a - 2c \\ 0 &= 2b - a \Rightarrow a = 2b \\ 1 &= 2c - b \end{aligned}\right\} \quad \left.\begin{aligned} 1 &= 4b - c \\ 1 &= 2c - b \end{aligned}\right\} \Rightarrow \left.\begin{aligned} b &= 2/3 \\ c &= 5/6 \end{aligned}\right\} a = 4/3$$

$\therefore \boxed{a = 4/3, \ b = 2/3, \ c = 5/6.}$