LECTURE 29: ORTHONORMAL SETS

Definition: We say that a subset S of a Hilbert (or inner product) space H is orthonormal if $\|x\| = 1 \ \forall x \in S$ and $(x,y) = 0 \ \forall x, y \in S$.

Exercise: If S is orthonormal (o.n.) then S is linearly independent.

Definition: S is a complete subset of H if $(x,y) = 0$ $\forall y \in S \Rightarrow x = 0$.

Examples (i) $l_2(\mathbb{R}) = H$ then $e_n = (0,0,0 \ldots, 0, 1,0,0\ldots.)$ (the 1 in the nth place) is a complete $o.n.$ set and $\forall x \in H \quad x = \sum_{j=1}^{\infty} (x,e_j)e_j$.

(ii) $L_2([0,\pi], \mathbb{C})$: $\left\{ \frac{1}{\sqrt{2\pi}} \exp(ijt) : j \in \mathbb{Z} \right\}$ is a $c.o.n.$ set.

(iii) $L_2([0,\pi], \mathbb{R})$: $\left\{ \frac{1}{\sqrt{2\pi}}, \frac{1}{\sqrt{\pi}} \cos(nt), \frac{1}{\sqrt{\pi}} \sin(nt) : n \in \mathbb{N} \right\}$

(iv) Let $D = \{z \in \mathbb{C} : |z| < 1\}$ and let H be the set of functions which are $\mathbb{C}$ analytic on D and satisfy

$$\iint_D |f(z)|^2 dxdy < \infty \quad \text{with}$$

$$(f,g) = \iint_D f\bar{g}\, dxdy.$$ Then a $c.o.n.$ set is defined by

$$\varphi_n(z) = \sqrt{\frac{n}{\pi}}\, z^{n-1}, \quad n = 1,2,3\ldots \quad .$$

We will not study these particular especially important spaces and $c.o.n.$ sets but develop the basic properties in general. In the following lecture we will apply example (iii).

Definition: We say (x,e_j) is the Fourier Coefficient of x with respect to e_j.

Theorem 61: [Bessel's Inequality.] If

$\{x_1, \ldots, x_n\}$ is $o.n.$ in an inner product space H

then

$$\sum_{j=1}^{n} |(x,x_j)|^2 \leq \|x\|^2$$

Proof:

$$0 \leq \|x - \sum_{j=1}^{n} (x,x_j)x_j\|^2$$

$$= (x - \sum_{j=1}^{n} (x,x_j)x_j,\ x - \sum_{j=1}^{n} (x,x_j)x_j)$$

$$= \|x\|^2 - \sum_{j=1}^{n} \overline{(x,x_j)}\,(x,x_j) - \sum_{j=1}^{n} (x,x_j)\,\overline{(x,x_j)}$$

$$+ \underbrace{\sum_{j=1}^{n} \sum_{i=1}^{n} (x,x_i)\,\overline{(x,x_j)}\,(x_i,x_j)}_{\sum_{j=1}^{n} (x,x_j)\,\overline{(x,x_j)}}$$

$$= \|x\|^2 - \sum_{j=1}^{n} |(x,x_j)|^2 \quad .$$

Theorem 62: Let $\{x_\alpha : \alpha \in A\}$ be an $o.n.$ set in an innerproduct space H. Then for each $x \in H$ at most a countable number of the $\{(x,x_\alpha) : \alpha \in A\}$ are different from 0.

Proof: Let

$$A_n = \{\alpha \in A : |(x,x_\alpha)| \geq 1/n\}$$

Then if A_0 is the set of non-zero Fourier coefficients of x

$$A_0 = \bigcup_{n=1}^{\infty} A_n .$$

If A_0 is uncountable then $\exists\, n_0 \in \mathbb{N}$ such that A_{n_0} is uncountable. Then $\exists\, \{\alpha_1, \alpha_2, \ldots\} \subset A_{n_0}$. By Bessel's inequality

$$\frac{N}{n_0^2} \leq \sum_{j=1}^{N} |(x, x_{\alpha_j})|^2 \leq \|x\|^2 .$$

Hence $\forall\, N \in \mathbb{N}$, $N \leq n_0^2 \|x\|^2$, a contradiction. Therefore A_0 must be countable.

Given $x \in H$ label the non zero (x, x_α) by $(x, x_{\alpha_1}), (x, x_{\alpha_2}), \ldots.$ and consider

$$\sum_{j=1}^{\infty} (x, x_{\alpha_j}) x_{\alpha_j} .$$

<u>Theorem 63</u>: If $\{x_i : i \in \mathbb{N}\}$ is an *o.n.* set in a Hilbert space H then

$\sum_{j=1}^{\infty} (x, x_j) x_j$ converges and the sum is independent of the order of summation.

<u>Proof</u>: Let $S_n = \sum_{j=1}^{n} (x, x_j) x_j$. Then

$$\begin{aligned} \|S_n - S_{m-1}\|^2 &= \Big\| \sum_{j=m}^{n} (x, x_j) x_j \Big\|^2 \\ &= \sum_{j=m}^{n} \|(x, x_j) x_j\|^2 \\ &= \sum_{j=m}^{n} |(x, x_j)|^2 \qquad \text{if } n \geq m \end{aligned}$$

By Bessel's inequality, $\forall\, N \; \sum_{j=1}^{N} |(x, x_j)|^2 \leq \|x\|^2$. Hence

$\sum_{j=1}^{\infty} |(x, x_j)|^2 \leq \|x\|^2$ (*) and the sum on the LHS converges absolutely in $\mathbb{R}$.

Hence given $\varepsilon > 0$ $\exists\, m - 1 \in \mathbb{N}$ such that

$$\sum_{m-1}^{\infty} |(x, x_j)|^2 < \varepsilon .$$

Hence $\|S_n - S_{m-1}\|^2 < \varepsilon \quad \forall\, n \geq m$. Therefore the sequence of partial sums is Cauchy. Since H is complete the sequence converges.

To prove that the limit is independent of the order use (*) and the fact that the sum of a convergent series of positive terms is independent of the order.

Given $x \in H$ we may now take the non-zero fourier coefficients and label them $\{(x,x_j) : j \in \mathbb{N}\}$, then form the sum

$$\sum_{j=1}^{\infty} (x,x_j)x_j \in H, \quad \text{and finally define}$$

$$\sum_{\alpha\in A} (x,x_\alpha)x_\alpha = \sum_{j=1}^{\infty} (x,x_j)x_j \; .$$

The equivalent identifying properties of complete $o.n.$ sets are set out in the following:

Theorem 64: Let $\{x_\alpha : \alpha \in A\}$ be $o.n.$ in A.

Then the following are equivalent.

(i) $\{x_\alpha\}$ is complete $((x_\alpha,x) = 0 \;\; \forall\, \alpha \Rightarrow x = 0)$,

(ii) $\forall\, x \in H, \;\; x = \sum_{\alpha\in A} (x,x_\alpha)x_\alpha$

(iii) $\forall\, x \in H, \;\; \|x\|^2 = \sum_{\alpha\in A} |(x,x_\alpha)|^2$

(iv) $\forall\, x,\, y \in H, \;\; (x,y) = \sum_{\alpha\in A} (x,x_\alpha)\overline{(y,y_\alpha)}$

Proof: (i) $\Rightarrow$ (ii) Let $y = \sum_{\alpha\in A} (x,x_\alpha)x_\alpha$

$$= \sum_{j=1}^{\infty} (x,x_{\alpha_j})x_{\alpha_j} \; .$$

Then $(y - x,x_\alpha) = (y,x_\alpha) - (x,x_\alpha)$

$$(y,x_\alpha) = \lim_{N\to\infty} \left(\sum_{j=1}^{N} (x,x_{\alpha_j})x_{\alpha_j}, x_\alpha \right).$$

$$= \lim_{N\to\infty} (x,x_\alpha) = (x,x_\alpha).$$

Thus $(y - x, x_\alpha) = 0 \ \forall \alpha \in A$ and so $y - x = 0$ or $y = x$.

(ii) $\Rightarrow$ (i): If $(x,x_\alpha) = 0 \ \forall \alpha \Rightarrow x = \sum_{\alpha\in A} 0.x_\alpha = 0$.

(iv) $\Rightarrow$ (iii): If $\forall x,y \ (x,y) = \sum_{\alpha\in A} (x,x_\alpha)\overline{(y,y_\alpha)}$

let $x = y \Rightarrow \|x\|^2 = \sum_{\alpha\in A} |(x,x\ )|^2$ i.e. (iii).

(ii) $\Rightarrow$ (iv): let $x = \sum_{\alpha\in A} (x,x_\alpha)x_\alpha = \sum_{j=1}^{\infty} (x,x_{\alpha_j})x_{\alpha_j}$ and let

$$y = \sum_{\alpha\in A} (y,x_\alpha)x_\alpha = \sum_{j=1}^{\infty} (y,x_{\alpha_j})x_{\alpha_j}.$$

Note that by suitable relabelling and inclusion of zero terms in both sums we can, and have, assumed matched labelling in these sums.

$$\text{Then} \quad (x,y) = \lim_{N\to\infty} \left(\sum_{j=1}^{N} (x,x_{\alpha_j})x_{\alpha_j}, \sum_{j=1}^{N} (y,x_{\alpha_j})x_{\alpha_j} \right)$$

$$= \sum_{j=1}^{\infty} (x,x_{\alpha_j})\overline{(y,y_{\alpha_j})}$$

$$= \sum_{\alpha\in A} (x,x_\alpha)\overline{(y,x_\alpha)}$$

(iii) $\Rightarrow$ (i) If $\|x\|^2 = \sum_{\alpha\in A} |(x,x_\alpha)|^2$ and $(x,x_\alpha) = 0 \ \forall \alpha$ then $\|x\|^2 = \sum_{\alpha\in A} 0^2 = 0 \Rightarrow x = 0$ hence (i).

<u>Notes</u>: (i) Normally $|A| = \aleph_0$ i.e. complete orthonormal sets are countable. In this case we say that H is <u>separable</u>. [A space (X,τ) is said to be <u>separable</u> if it contains a countable dense subset. One can show that H is separable $\Longleftrightarrow$ it is separable in its induced topology].

Spaces (i) → (iv) given at the beginning of the lecture are all separable.

(ii) Normally the $\{x_n\}$ are nice functions (say infinitely differentiable if H is $L_2[0,1]$).

(iii) Often a *c.o.n.* set is formed from a desirable set of functions which is complete but not *o.n.* For example $\{1,x,x^2 ..\}$ in $C[0,1] \subset L_2[0,1]$, We manufacture an *o.n.* set from a linearly independent set using the so called Gram-Schmitt procedure:

Let $y_1 = x_1$

$$y_2 = x_2 - (x_2,y_1)y_1/\|y_1\|^2$$

$$\vdots$$

Then let $w_n = y_n/\|y_n\|$ $\forall\, n \in \mathbb{N}$. The space spanned by the $\{x_n\}$ = the space spanned by the $\{w_n\}$ and the $\{w_n\}$ is *o.n.*

(iv) The following exercise is relevant to this discussion: a linearly independent set is complete $\Longleftrightarrow$ its linear span is dense in H.

"Far better an approximate answer to the right question, which is often vague, than an exact answer to the wrong question, which can always be made precise."

(J.W. Tukey)