The University of Waikato Department of Mathematics

Elements of Analysis Tutorial for 30th September: math252-10B

Uniform convergence of sequences and series of functions:

1. For each $n \in \mathbb{N}$ and $x \in \mathbb{R}$ let

$$f_n(x) = \frac{x}{n} + x.$$

Show that the sequence of functions $(f_n : n \in \mathbb{N})$ converges pointwise to the function $h(x) = x$ on $\mathbb{R}$.

2. Now show the convergence of the sequence in 1. is uniform on $[0,1]$ and then $[0,2]$

3. Keeping to this same sequence show the convergence is uniform on $[-c, c]$ for every real $c > 0$.

4. Now verify each f_n is continuous and deduce h is continuous on $\mathbb{R}$, even though the convergence is not uniform on $\mathbb{R}$. To show it is not uniform try $\epsilon = 1$ and test $|f_n(x) - h(x)| < 1$ at $x = N_\epsilon + 1$.

5. Show that $f_n' \to h_1$, where $h_1(x) = 1$ for all x, and that the convergence is uniform on $[0,2]$. Hence verify that $\lim_{n\to\infty} f_n' = (\lim_{n\to\infty} f_n)'$. Then show this explicitly.

6. Let $g_n(x) = nx$ for $0 \leq x \leq 1/n$ and $g_n(x) = n(1-x)/(n-1)$ for $n \in \mathbb{N}$ and $x \in [0,1]$. Verify pointwise convergence to h_0, where $h_0(x) = 0$ for all x, and then show, by explicit integration of g_n that this convergence is not uniform. Draw some pictures then to show what is going on.

30th September 2010