The University of Waikato Department of Mathematics Elements of Analysis Tutorial 5 and Assignment 4: math252-10B

For the Tutorial on 23rd September, attempt questions 1,2, and 3. Assignment 4 due Thursday 30th September is 4,5,6.

1. Let $S, T \subset \mathbb{R}$ be a bounded subsets. Prove that $\sup S+T \leq \sup S+\sup T$.

2. An indefinite integral for

$$\int_0^b \frac{\sin x}{x} dx$$

does not exist in "elementary terms" and you don't want to use a numerical method because $b > 0$ needs to remain as a parameter. Use the Taylor expansion of $\sin x$ to get polynomial approximations to the integrand of degree 3 and degree 5 and so find functions $f(x),\ g(x)$ such that

$$f(b) \leq \int_0^b \frac{\sin x}{x} dx \leq g(b)$$

for $0 \leq b \leq \pi/2$.

3. Let $f(x) = x^2$ on $[0, 4]$. Given $\varepsilon > 0$ find a $\delta_\varepsilon > 0$, independent of $a \in [0, 4]$ such that

$$|x - a| < \delta_\varepsilon \implies |f(x) - f(a)| < \varepsilon.$$

Hint: use the Mean Value Theorem.

4. Define $f : [0, 2] \to \mathbb{R}$ by $f(x) = x$ if $x \leq 1$ and $f(x) = 2x$ if $x > 1$. Evaluate $F(x) = \int_0^x f$ as an explicit function of x and show that this function is continuous at $x = 1$. Verify that $F'(x) = f(x)$ when $x \neq 1$. What can you say about $F'(1)$.

5. Show that $g(x) = 1/x$ is NOT uniformly continuous on on $(0, 1]$ by showing that for $\varepsilon_0 = 1$ there is no $n \in \mathbb{N}$ such that $\delta = 1/n$ works everywhere. I.e find two points $x_n,\ y_n$ for which it fails.

6. Towards a proof that the product of two Riemann integrable functions is Riemann integrable. First show that you need only consider squares of functions since

$$f.g = \frac{1}{4}((f + g)^2 - (f - g)^2).$$

Show then that you need consider only positive valued functions because $f(x).f(x) = |f(x)|^2$. Then, if $0 \leq f(x) \leq M$ on $[a, b]$ show $f^2(x) - f^2(y) \leq 2M(f(x) - f(y))$. Now suggest how the proof might be completed.

23rd September 2010