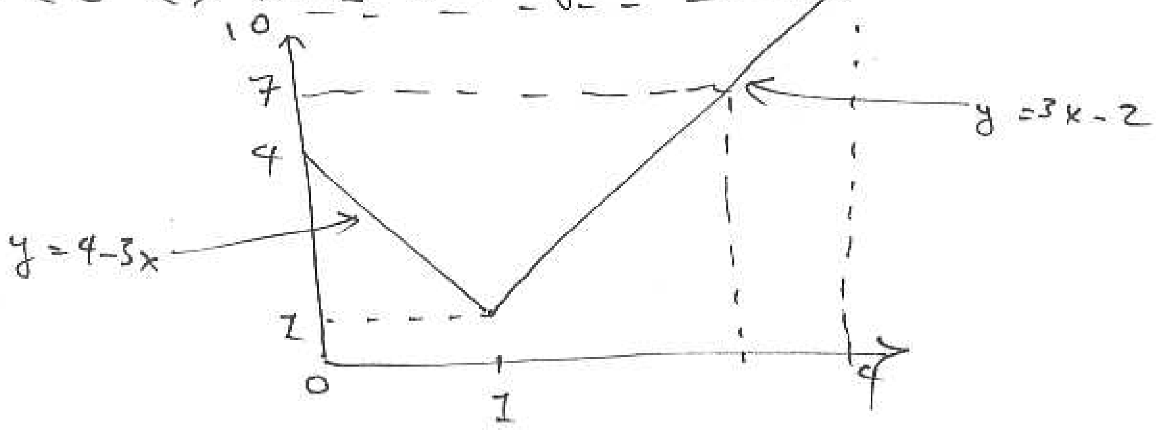
mah252-09B Test 2 Solutions (rJ1)

(1) $f(x) = 3|x-1|+1$ on $[0,4]$

If $x-1 \geq 0 \Leftrightarrow x \geq 1$ then $f(x) = 3(x-1)+1 = 3x-2$

If $x-1 < 0 \Leftrightarrow x < 1$ then $f(x) = 3(1-x)+1 = 4-3x$



$f(0) = 4-3.0 = 4$, $f(1) = 3|1-1|+1 = 1$, $f(4) = 3.4-2 = 10$

Hence

$m = 1 \leq f(x) \leq 10 = M$ on $[0,4]$ or

$m = 1 \leq f(x) \leq 7 = M$ on $[0,3]$

Solve $7 = f(\xi) = 3\xi - 2 \Rightarrow 3\xi = 9 \Rightarrow \underline{\xi = 3}$.

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(2) $f(0+h) = f(0) + f'(0)h + \dfrac{f''(0+\theta h)}{2!}h^2$

$f(x) = x^3 + 2x^2$

$f'(x) = 3x^2 + 4x$

$f''(x) = 6x + 4$

$h^3 + 2h^2 = f(h) = 0 + 0.h + \frac{1}{2}(6\theta h + 4)h^2$

$\Rightarrow \cancel{h^2}(h+2) = \frac{h^2}{2}(6\theta h + 4) = \cancel{h^2}(3\theta h + 2)$

$\Rightarrow h + 2 = 3\theta h + 2 \Rightarrow \theta = \frac{1}{3}$

$f'(0) = 0$ & $f''(0) = 4 > 0 \Rightarrow (0, f(0))$ is a local min.

Inflection $f''(\xi) = 0 \Rightarrow 0 = 6\xi + 4 \Rightarrow \xi = \frac{-2}{3}$

so $\left(-\frac{2}{3}, f(-\frac{2}{3})\right)$ is a point of inflection.

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(3) Def^n $U(f,P) = \sum_{i=1}^{n} M_i(f)\Delta x_i$ where

$M_i(f) = \sup\{f(x) : x_{i-1} \le x \le x_i\}$, $\Delta x_i = x_i - x_{i-1}$,

$P = (x_0, x_1, \ldots, x_n)$ $x_0 = a$, $x_n = b$.

By "f is Ri. Int. on $[a,b]$" we mean $\sup_P L(f,P) = \inf_P U(f,P)$

and then $\int_a^b f$ is the common value.

If $f(x) \ge 0 \Rightarrow \inf\{f(x) : x_{i-1} \le x \le x_i\} \ge 0$

$\Rightarrow L(f,P) \ge 0 \ \forall P \Rightarrow \sup L(f,P) \ge 0 \Rightarrow \int_a^b f \ge 0$.

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(4) By "f is uniformly continuous on $[a,b]$" we mean $\forall \varepsilon > 0\ \exists \delta_\varepsilon > 0$ such that $\forall x, y \in [a,b]$, $|x-y| < \delta_\varepsilon \Rightarrow |f(x) - f(y)| < \varepsilon$.

If f is continuous but ~~not~~ uniformly continuous $\exists \varepsilon_0 > 0$ so.... , $\boxed{\text{see notes}}$

Let $b > 0$ be fixed. By the M.V.Th. $f(x) - f(y) = f'(\xi)(x-y)$

so $|f(x) - f(y)| = |4\xi| \cdot |x-y| \le 4b|x-y|$ since $x < \xi < y \Rightarrow \xi \le b$.

Hence if $\varepsilon > 0$ is given & we let $\delta_\varepsilon = \frac{\varepsilon}{4b}$, if $|x-y| < \delta_\varepsilon \Rightarrow$

$|f(x) - f(y)| \le 4b\,\delta_\varepsilon = 4b\frac{\varepsilon}{4b} = \varepsilon \Rightarrow f$ is unif. cont on $[0,b]$

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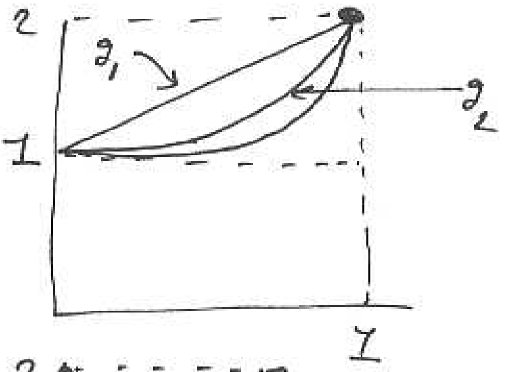
(5) If $f_n \to f$ uniformly let $\xi \in [a,b]$ etc $\boxed{\text{see notes}}$

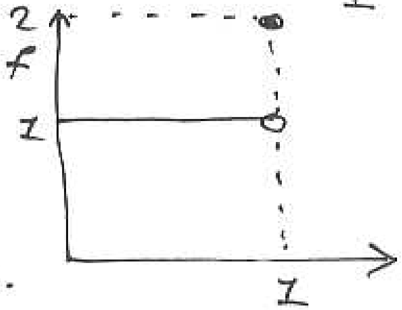
If $g_n(x) = x^n + 1$ then if $0 \le x < 1$

$\lim_{n\to\infty} g_n(x) = \lim_{x\to\infty}(x^n + 1) = \lim_{n\to\infty} x^n + 1$

$= 0 + 1 = 1$

& $\lim_{n\to\infty} g_n(1) = \lim_{n\to\infty}(1^n + 1) = 2$





If. $f(x)$ is the pointwise limit of (g_n)

then $f(x)$ is discontinuous (at $x = 1$).

Since each $g_n(x)$ is continuous, if the convergence was uniform then the limit fun. would be continuous. But it is not. $\therefore$ the convergence is not unif.

It is unif on $[0,c]$ $\forall\ 0 < c < 1$. //