UNIVERSITY OF WAIKATO
Department of Mathematics

MATH252-09B – Elements of Analysis

TEST 2

Wednesday 7 October 2009 - (55 mins) – Answer **ALL** questions

---

1. Let $f(x) = 3|x-1|+1$ on $[0,4]$. Sketch the graph of $y = f(x)$. Find bounds m and M such that $m \le f(x) \le M$ for all x in $[0,3]$. Choose a number $c \in (m, M)$ and solve $f(\xi) = c$ for ξ.

2. Give the general form of the Taylor expansion for a function about $a = 0$ with 2 terms and a remainder of order 2 in the form $f(0+h) = \boxed{} + \boxed{} + \boxed{}$.

   Now let $f(x) = x^2(x+2)$ and for given h, find the corresponding value of θ. Then find the nature of the critical point of $y = f(x)$ at $x = 0$ and the coordinates of the point of inflection.

3. Let $f:[a,b] \to \mathbb{R}$ be a bounded function and let P be a partition of $[a,b]$. Give a definition of the upper sum $U(f,P)$ and explain the meanings of "f is Riemann Integrable on $[a,b]$" and $\int_a^b f$. Let $f(x)$ be Riemann Integrable on $[a,b]$ and suppose $f(x) \ge 0$ for $a \le x \le b$. Prove that $\int_a^b f \ge 0$.

4. Define the expression "$f:[a,b] \to \mathbb{R}$ is uniformly continuous". Assuming every bounded sequence of real numbers has a convergent subsequence, prove that every continuous function $f:[a,b] \to \mathbb{R}$ is uniformly continuous.

   Given $b > 0$ and $f:[0,b] \to \mathbb{R}$ is $f(x) = 2x^2+1$, use the Mean Value Theorem to show directly that f is uniformly continuous.

5. Let (f_n) be a sequence of continuous functions on $[a,b]$ such that $f_n \to f$ uniformly on $[a,b]$. Show that $f(x)$ is a continuous function.

   Now let $g_n(x) = x^n + 1$ for $n = 1, 2, \ldots$. Show, by finding the pointwise limit, that (g_n) does **not** converge uniformly to its limit on $[0,1]$. Find a subinterval of $[0,1]$ on which it does converge uniformly.