Math 252A-07 Elements of Analysis and Algebra

Test I Solutions

1 (a) $S = \left\{ x \in \mathbb{R} : \dfrac{x+1}{x-1} > 2 \right\}$.

Firstly, $1 \notin S$ since $x = 1 \Rightarrow x-1 = 0$ & $\div$ by 0 is not allowed.

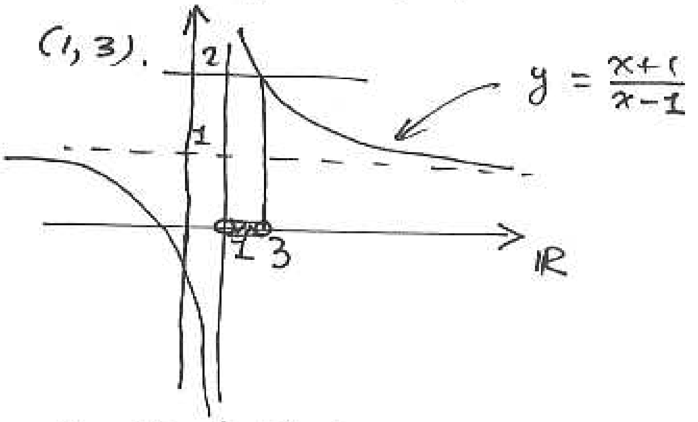
If $x-1>0$ then $\dfrac{x+1}{x-1} > 2 \iff x+1 > 2(x-1)$
($\iff x>1$) $\iff x+1 > 2x-2$
$\iff 3 > x$

Hence $1<x<3$ is in S or $(1,3) \subset S$.

If $x-1<0$ then $\dfrac{x+1}{x-1} > 2 \iff x+1 < 2(x-1)$
($\iff x<1$) $\iff 3 < x$

But $x<1$ and $3<x$ is the empty set $\emptyset$.

Hence the set of points is $(1,3)$.



(b) $|x+2| = |x-3+5|$
$\leq |x-3| + |5|$ by the Δ law.
$= |x-3| + 5$
$< 1 + 5 = 6$

2 (a) By $\lim_{n\to\infty} a_n = L$ we mean $\forall \varepsilon>0, \exists N_\varepsilon \in \mathbb{N}$ so that $\forall n \geq N_\varepsilon$
$|a_n - L| < \varepsilon$.

(b) By $\lim_{n\to\infty} a_n = \infty$ we mean $\forall M>0\ \exists N_M \in \mathbb{N}$ so that $\forall n \geq N_M$
$a_n > M$.

(c) By $\sum_{n=1}^{\infty} a_n = S$ we mean that if $S_n = a_1 + a_2 + \cdots + a_n$, $n \geq 1$
$S = \lim_{n\to\infty} S_n$.

(d) By $\alpha = \operatorname{lub}(S)$ we mean (i) $\forall x \in S$, $x \leq \alpha$, and
(ii) $\forall \varepsilon>0, \exists x \in S$ so $\alpha - \varepsilon < x \leq \alpha$.

(3) (a) $\lim_{n\to\infty} \frac{3n^3+n^2+1}{2n^3+n+1} = \lim_{n\to\infty} \frac{3+\frac{1}{n}+\frac{1}{n^3}}{2+\frac{1}{n^2}+\frac{1}{n^3}}$

$= \frac{3}{2}.$

(b) $\lim_{n\to\infty} \frac{\pm 1}{n^2+1} = 0$ & $-1 \le \sin(n) \le 1 \Rightarrow$

$-\frac{1}{n^2+1} \le \frac{\sin(n)}{n^2+1} \le \frac{1}{n^2+1}$

$\Rightarrow \lim_{n\to\infty} \frac{\sin(n)}{n^2+1} = 0.$

(c) $\frac{1}{3^n} + \frac{1}{n} \geq \frac{1}{n}$ and $\sum_{n=1}^{\infty} \frac{1}{n} = \infty \Rightarrow \sum_{n=1}^{\infty} \left(\frac{1}{3^n} + \frac{1}{n}\right) = \infty.$

(d) Let $b_n = \frac{1}{n}$ and $s_n = \sum_{j=1}^{n} \frac{(-1)^{j+1}}{j}$. Then $|s - s_n| \le b_{n+1}$

so we can estimate the sum as $b_1 - b_2 + b_3 - b_4 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4}$

$= \frac{12-6+4-3}{12}$

$= \frac{7}{12} = .58\dot{3}$

with error less than $\frac{1}{5} = b_5$

Note: the exact sum is $\ln(2) = 0.693147\ldots$ and differs less than $0.1098 < b_5 = 0.2$. //

(4) Given $\varepsilon > 0$, want $|a_n - 1| < \varepsilon \iff \left|\frac{n-1}{n+1} - 1\right| < \varepsilon$

$\iff \left|\frac{n-1-n-1}{|n+1|}\right| < \varepsilon$

$\iff \frac{2}{n+1} < \varepsilon$

$\iff \frac{2}{\varepsilon} - 1 < n$

So let N_ε be any element of $\mathbb{N}$ with $\frac{2}{\varepsilon} - 1 < N_\varepsilon$. Then if $n \geq N_\varepsilon$ we get $n \geq N_\varepsilon > \frac{2}{\varepsilon} - 1$ so $|a_n - 1| < \varepsilon$. $\therefore \lim_{n\to\infty} a_n = 1$.

By limit theorems $\lim_{n\to\infty} \frac{n-1}{n+1} = \lim_{n\to\infty} \frac{1-\frac{1}{n}}{1+\frac{1}{n}} = \frac{\lim_{n\to\infty}(1-\frac{1}{n})}{\lim_{n\to\infty}(1+\frac{1}{n})}$

$= \frac{\lim_{n\to\infty} 1 - \lim_{n\to\infty} \frac{1}{n}}{\lim_{n\to\infty} 1 + \lim_{n\to\infty} \frac{1}{n}}$

$= \frac{1-0}{1+0} = 1$

since the lt. of a quotient is the quotient of the limits, the lt. of a sum is the sum of the limits, a constant seq. tends to the constant value, and $1/n \to 0$ as $n \to \infty$.

(a) Abel's Divergence Test : Firstly, if $a_n = \frac{3}{4n+1}$,

then (a_n) is decreasing : $a_{n+1} \le a_n \Leftrightarrow \frac{3}{4(n+1)+1} \le \frac{3}{4n+1}$

$\Leftrightarrow \frac{1}{4n+5} \le \frac{1}{4n+1}$

$\Leftrightarrow 4n+5 \geq 4n+1$

$\Leftrightarrow 5 \geq 1$ which is true.

Secondly $na_n \not\to 0$: $\lim_{n\to\infty} na_n = \lim_{n\to\infty} \frac{3n}{4n+1} = \lim_{n\to\infty} \frac{3}{4+\frac{1}{n}}$

$= \frac{3}{4} \neq 0.$

Hence the series $\sum_{n=1}^{\infty} a_n$ $\mathcal{D}$.

By the Integral Test $f(x) = \frac{3}{4x+1}$: $\int_1^T f(x)dx = 3\int_1^T \frac{dx}{4x+1}$

$= \frac{3}{4}\ln(4x+1)\Big|_1^T$

$= \frac{3}{4}\ln(4T+1) - \frac{3}{4}\ln(5)$

$\to \infty$ as $T \to \infty$

Hence $\sum_1^{\infty} a_n$ $\mathcal{D}$.

(b) Firstly if $a_n = \frac{1}{2^n+n}$ and $b_n = \frac{1}{2^n}$, $0 < a_n < b_n \ \forall n \in \mathbb{N}$

Secondly $\frac{b_{n+1}}{b_n} = \frac{1/2^{n+1}}{1/2^n} = \frac{2^n}{2^{n+1}} = \frac{1}{2} < 1 \ \forall n \in \mathbb{N}$. Hence,

by D'Alembert's Test, $\sum_1^{\infty} b_n$ $\mathcal{C}$, Thus by the Comparison Test $\sum_1 a_n$ does also.

(c) If $b_n = \frac{1}{\sqrt{n}}$, $\lim_{n\to\infty} b_n = 0$. Also $b_{n+1} \le b_n \Leftrightarrow$

$\frac{1}{\sqrt{n+1}} \le \frac{1}{\sqrt{n}} \Leftrightarrow \frac{1}{n+1} \le \frac{1}{n}$

which is true.

Hence, by the Alternating Series Test $\sum_1^{\infty} (-1)^{n+1} b_n$ $\mathcal{C}$.

But $\left|(-1)^{n+1} b_n\right| = b_n$. If $f(x) = \frac{1}{\sqrt{x}}$, $\int_1^T f(x)dx = 2\sqrt{T} - 2 \to \infty$

Hence $\sum_1^{\infty} b_n$ $\mathcal{D}$ so $\sum_1^{\infty} (-1)^{n+1}/\sqrt{n}$ is convergent, but not Absolutely Convergent i.e. its conditionally convergent.