UNIVERSITY OF WAIKATO
Department of Mathematics

MATH252-10B – Elements of Analysis

TEST 2

Thursday 7 October 2010 - (55 mins) – Answer **ALL** questions

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1. Let $f(x)=|x-1|+2$ on $[0,3]$. Sketch the graph of $y=f(x)$. Find bounds m and M such that $m \leq f(x) \leq M$ for all x in $x=y$, Choose a number $c \in (m, M)$ and solve $f(\xi)=c$ for ξ.

2. Give the general form of the Taylor expansion for a function about $a=0$ with 2 terms and a remainder of order 2 in the form $f(0+h)=\boxed{}+\boxed{}+\boxed{}$.

   Now let $f(x)=x^2(x+1)$ and for given h, find the corresponding value of θ. Then find the nature of the critical point of $y=f(x)$ at $x=0$ and the coordinates of the point of inflection.

3. Let $f:[a,b] \to \mathbb{R}$ be a bounded function and let P be a partition of $[a,b]$. Give a definition of the lower sum $L(f,P)$ and explain the meanings of "f is Riemann Integrable on $[a,b]$" and $\int_a^b f$. Let $f:[0,2] \to \mathbb{R}$ satisfy $f(x)=0$ for $x \neq 1$ and $f(x)=1$. Prove that $\int_a^b f = 0$.

4. Define the expression "$f:[a,b] \to \mathbb{R}$ is uniformly continuous". Assuming every bounded sequence of real numbers has a convergent subsequence, prove that every continuous function $f:[a,b] \to \mathbb{R}$ is uniformly continuous.

   Given $b>0$ and $f:[0,b] \to \mathbb{R}$ is $f(x)=x^2+2x$, use the Mean Value Theorem to show directly, without using the definition, that f is uniformly continuous on $[0,b]$.

5. Let $f:[a,b] \to \mathbb{R}$ be integrable. Prove that if we set $F(y)=\int_a^y f$, $a<y<b$, then $F(y)$ is continuous on (a,b). Then show that if $a<y<b$ and $f(x)$ is continuous at $x=y$, then $F'(y)=f(y)$.

   Illustrate the result when $$f(x)=\begin{cases} x, & 0 \leq x \leq 1 \\ 2, & 1 \leq x \leq 2 \end{cases}.$$

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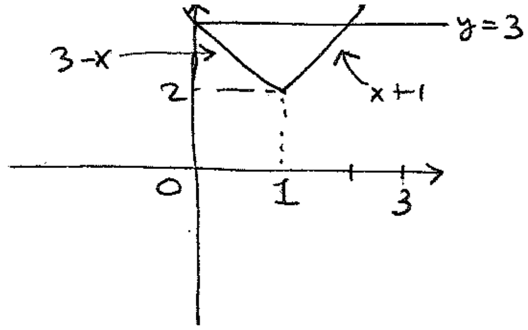


(1) $f(x) = |x-1| + 2$

"shift $y=|x|$ to $x=1$ and then move the graph up by 2 units" $\longrightarrow$

OR $x-1 \geq 0 \Rightarrow f(x) = x-1+2 = x+1$

$x-1<0 \Leftrightarrow x<1 \Rightarrow f(x) = -(x-1)+2 = 3-x$

then $f(3) = |3-1|+2 = 4$

& $2 = m \leq f(x) \leq 4 \quad \forall x \in [0,3]$

Let $c=3$: $3 = f(\xi)$ & $\xi > 1 \Rightarrow 3 = \xi + 1 \Rightarrow \xi = 2$. //

(2) $f(0+h) = f(0) + f'(0)h + \dfrac{f''(0+\theta h)}{2!}h^2$

$$\left.\begin{array}{ll} f(x) = x^3+x^2 \Rightarrow & f(0)=0 \\ f'(x) = 3x^2+2x & f'(0)=0 \\ f''(x) = 6x+2 & f''(\theta h) = 6\theta h+2 \end{array}\right\} \Rightarrow$$

$h^3 + h^2 = 0 + 0 + \frac{1}{2}(6\theta h+2)h^2$

$\cancel{h^3} + \cancel{h^2} = 3\theta \cancel{h^3} + \cancel{h^2} \quad \Rightarrow \quad \theta = 1/3$

At $x=0$ $f''(0) = 2 > 0$ & $f'(0)=0$ $\therefore$ local min. OR

$f(x) \approx x^2(0+1) = x^2$ so its like $y=x^2$ which has a local min.

For a pt of infle $f^{(2n+1)}(a) \neq 0$ but $f''(a) = \ldots f^{(2n)}(a) = 0$

Here $f''(x) = 6x+2 \Rightarrow 0 = f''(a) = 6a+2 \Rightarrow a = -1/3$

& $f'''(-1/3) = 2 \neq 0$ $\therefore$ $x = -1/3$ give a pt. of infle.

(3) $L(f,P) = \sum_{j=1}^{n} m_j(f)\Delta x_j$ where $m_j(f) = \inf\{f(x) : x_{j-1} \leq x \leq x_j\}$

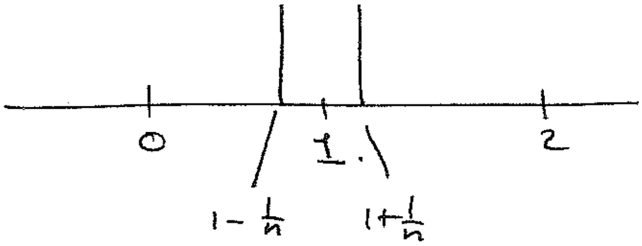
& $P = (x_0, \ldots, x_n)$

We say f is integrable on $[a,b]$ if.

$\sup_P (L(f,P)) = \inf_P (U(f,P))$ & write $\int_a^b f$

as the common value.

coords $\boxed{\left(-1/3, \frac{2}{27}\right)}$



Let $P_n = \left(0,\ 1-\frac{1}{n},\ 1+\frac{1}{n},\ 2\right)$

Then $L(f,P_n) = 0+0+0$

$U(f,P_n) = 0 + 1\cdot\frac{2}{n} + 0 = \frac{2}{n}$

$\therefore\quad U(f,P_n) - L(f,P_n) = \frac{2}{n} < \varepsilon \quad \forall n > \frac{2}{\varepsilon}$ so f is Ri. In.

& $\quad 0 = L(f,P_n) \le \int_0^2 f \le U(f,P_n) = \frac{2}{n} \quad \forall n \Rightarrow$

by squeezing, $\int_0^2 f = 0$.

(4) $f:[a,b]\to\mathbb{R}$ is uniformly continuous if $\forall\varepsilon>0$ $\exists\delta_\varepsilon>0$ such that $\forall x,y\in[a,b]$, $|x-y|<\delta_\varepsilon \Rightarrow |f(x)-f(y)|<\varepsilon$.

Let $\varepsilon>0$ be given & let $f:[a,b]\to\mathbb{R}$ be cont. but not unif cont (??).

Then $\exists\varepsilon_0>0$ such that $\forall\delta>0$ $\exists x_\delta, y_\delta$ so $|x_\delta-y_\delta|<\delta$ but $|f(x_\delta)-f(y_\delta)|\ge\varepsilon_0$

$\Rightarrow \forall n\in\mathbb{N}$ $\exists x_n, y_n$ so $|x_n-y_n|<\frac{1}{n}$ but $|f(x_n)-f(y_n)|\ge\varepsilon_0$.

$\exists$ subseq $x_{n_i}\to x$ in $[a,b]$ and $\exists$ subseq $y_{n_{i_j}}\to y$ in $[a,b]$ so also $x_{n_{i_j}}\to x$ in $[a,b]$

But $n_i \ge i$ and $n_{i_j}\ge i_j\ge j \Rightarrow |x_{n_{i_j}}-y_{n_{i_j}}|\le\frac{1}{j}$.

Let $j\to\infty$ & get $|x-y|\le 0 \Rightarrow x=y$.

But f is continuous at $x=y$ & thus $|f(x_{n_{i_j}})-f(y_{n_{i_j}})|\ge\varepsilon_0$

$\Rightarrow |f(x)-f(y)|\ge\varepsilon_0$

$\Rightarrow |f(x)-f(x)|\ge\varepsilon_0>0$

or $0>0$ (!!).

Hence f is unif cont. on $[a,b]$

Let $b>0$ $\quad f'(x) = 2x+1$. By the M.V.Th. $(f(x)-f(y)) = f'(\xi)(x-y)$

$\Rightarrow |f(x)-f(y)| = |f'(\xi)||x-y| = (2\xi+1)|x-y| \le (2b+1)|x-y|$

So we can, given $\varepsilon>0$, let $\delta_\varepsilon = \frac{\varepsilon}{2b+1}$. Then

$|x-y|<\delta_\varepsilon \Rightarrow |f(x)-f(y)| < \frac{(2b+1)\varepsilon}{2b+1} = \varepsilon$ $\therefore$ f is unif cont.

(5)

$$F(y) = \int_a^y f \qquad \text{Let} \quad h>0 \quad \& \quad |f(x)| \le M \ \forall x \in [a,b].$$

$$F(y+h) - F(y) = \int_a^{y+h} f - \int_a^y f = \int_y^{y+h} f$$

Thus $\displaystyle |F(y+h) - F(y)| = \left|\int_y^{y+h} f\right| \le \int_y^{y+h} |f| \le M\int_y^{y+h} 1 = Mh \le M|h|$

& the same is true if $h<0$. Hence F is (uniformly) cont on $[a,b]$.

If $a<y<b$ & once $\varepsilon>0$ is given, $\exists\, \delta_\varepsilon > 0$ such that

$|t-y| < \delta_\varepsilon \Rightarrow |f(t) - f(y)| < \varepsilon$. Let $h>0$ & consider

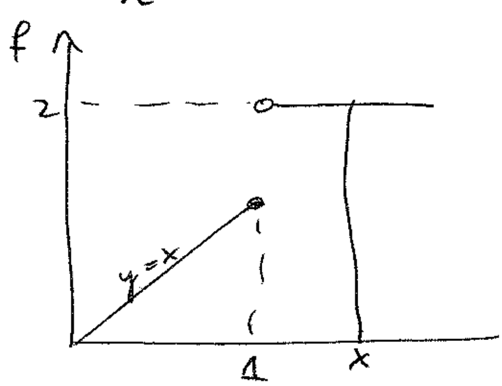
$$\left|\frac{F(y+h)-F(y)}{h} - f(y)\right| = \left|\frac{\int_y^{y+h} f - f(y)\int_y^{y+h} 1}{h}\right| = \left|\frac{\int_y^{y+h}(f(t) - f(y))\,dt}{h}\right|$$

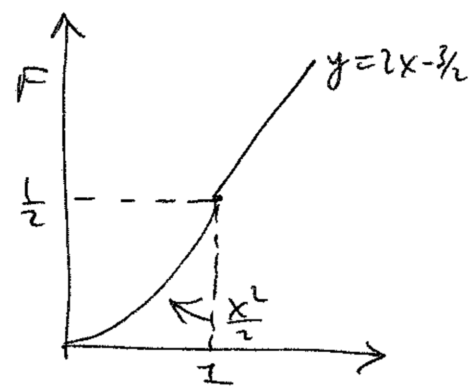
$$\le \frac{\int_y^{y+h} |f(t) - f(y)|\,dt}{|h|} < \frac{\varepsilon\int_y^{y+h} 1}{|h|} = \varepsilon$$

again the same is true for $y<0$

Hence $\displaystyle \lim_{h\to 0} \frac{F(y+h)-F(y)}{h} = f(y)$ so $F'(y) = f(y)$.

Illustration





$F(x) = \frac{x^2}{2} \qquad 0 \le x \le 1$

$\qquad = \frac{1}{2} + 2(x-1) = 2x - \frac{3}{2}$

& F is cont on $[0,2]$

& diff when $f(x)$ is continuous i.e. $(0,2)\setminus\{1\}$.