1 (a) $\lim_{n\to\infty} a_n = L$ means $\forall \varepsilon > 0 \ \exists N_\varepsilon \in \mathbb{N}$ so

n large enough $\forall n \geq N_\varepsilon$, $|a_n - L| < \varepsilon$. an close to L

(b) Abel's Test If $a_n \geq 0 \ \forall n$ and $a_{n+1} \leq a_n \ \forall n \geq 1$

and $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} n a_n = 0$. Thus,

if. $\lim_{n\to\infty} n a_n \neq 0$ then $\sum_{n=1}^{\infty} a_n$ diverges (to ∞).

(c) If there exists a function $f: [1,\infty) \to \mathbb{R}$ such that

$x < y \Rightarrow f(x) \geq f(y)$ and $\int_1^\infty f(x)\,dx \leq B$ then

if $a_n = f(n) \ \forall n \in \mathbb{N}$, $\sum_{n=1}^{\infty} a_n$ converges.

(d) $\lim_{x\to a+} f(x) = \infty$ means $\forall M > 0$ (large) $\exists \delta_M > 0$ (small) so

$a < x < a + \delta_M \Rightarrow f(x) > M$

x close to a & on the right f(x) large

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2 Let $S \subset \mathbb{R}$ be non-empty and bounded above. Then S has a least upper bound i.e. if $\exists B \in \mathbb{R}$ so $\forall x \in S$, $x \leq B$, then $\exists \beta \in \mathbb{R}$ so ① $\forall x \in S$ $x \leq \beta$ & ② $\forall \varepsilon > 0$ $\exists y \in S$ so $\beta - \varepsilon < y \leq \beta$.

If $0 < c < 1$ let $a_n = c^n$, $n \in \mathbb{N}$. Then $a_{n+1} = c^{n+1} = c.c^n = c.a_n$

$\Rightarrow \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} c.a_n = c \lim_{n\to\infty} a_n$ (*) Since (a_n) is decreasing $(a_{n+1} < a_n)$ and bounded below $(0 < a_n) \Rightarrow \lim_{n\to\infty} a_n$ exists. Let it be L. Then $L = \lim_{n\to\infty} a_{n+1} = cL \Rightarrow L = cL \Rightarrow L(1-c) = 0$

$\Rightarrow L = 0$ //

(3) Let $\varepsilon > 0$ be given.

Since $\lim_{x \to a+} f(x) = L$, $\exists\, \delta_1 > 0$ so $a < x < a+\delta_1 \Rightarrow |f(x) - L| < \frac{\varepsilon}{2}$

Since $\lim_{x \to a+} g(x) = M$, $\exists\, \delta_2 > 0$ so $a < x < a+\delta_2 \Rightarrow |f(x) - M| < \frac{\varepsilon}{2}$ $\therefore$

Let $\boxed{\delta_\varepsilon = \min\{\delta_1, \delta_2\}}$. If. $a < x < a + \delta_\varepsilon$ then $a < x < a+\delta_1$ and $a < x < a+\delta_2$

so
$$\begin{aligned}
|(f(x) + g(x)) - (L+M)| &= |(f(x) - L) + (g(x) - M)| \\
&\underset{\Delta}{\leq} |f(x) - L| + |g(x) - M| \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon
\end{aligned}$$

Therefore $\lim_{x \to a+} f(x) + g(x) = L + M$ //

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(4) (a) Let $\varepsilon > 0$ be given. Working back we want

$$\begin{aligned}
\left| \frac{4n+1}{2n+1} - 2 \right| < \varepsilon \quad &\Leftrightarrow \left| \frac{4n+1-(4n+2)}{2n+1} \right| < \varepsilon \\
&\Leftrightarrow \frac{|-1|}{|2n+1|} < \varepsilon \Leftrightarrow \frac{1}{2n+1} < \varepsilon \\
&\Leftrightarrow 2n+1 > \frac{1}{\varepsilon} \quad (\text{since } \varepsilon > 0 \text{ \& } n \geq 1) \\
&\Leftrightarrow n > \frac{1}{2\varepsilon} - \frac{1}{2}
\end{aligned}$$

so let $\boxed{N_\varepsilon > \frac{1}{2\varepsilon} - \frac{1}{2}}$. Then if $n \geq N_\varepsilon \Rightarrow n > \frac{1}{2\varepsilon} - \frac{1}{2}$

then $\left| \frac{4n+1}{2n+1} - 2 \right| < \varepsilon$. Hence $\lim_{n \to \infty} \frac{4n+1}{2n+1} = 2$.

OR $\lim_{n \to \infty} \frac{4n+1}{2n+1} = \lim_{n \to \infty} \frac{4 + \frac{1}{n}}{2 + \frac{1}{n}} = \frac{4+0}{2+0} = 2$. //

(b) $a_n = \frac{2^n}{n!} \Rightarrow \frac{a_{n+1}}{a_n} = \frac{2^{n+1}}{(n+1)n!} \times \frac{n!}{2^n} = \frac{2}{n+1}$

& $\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = \lim_{n \to \infty} \frac{2}{n+1} = 0 < 1 \Rightarrow \sum_{n=1}^{\infty} a_n$ converges.

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(5) $f(x) = 2x + \frac{1}{x}$, $f(1) = 2 + \frac{1}{1} = 3$

Let $\varepsilon > 0$ be given. We want $|f(x) - f(1)| < \varepsilon$

$\Leftrightarrow \left| 2x + \frac{1}{x} - 3 \right| < \varepsilon$

$\Leftrightarrow \left| \frac{2x^2 + 1 - 3x}{x} \right| < \varepsilon \Leftrightarrow \left| \frac{(2x-1)(x-1)}{x} \right| < \varepsilon$

$\Leftrightarrow \frac{|2x-1||x-1|}{|x|} < \varepsilon$.

Let $\delta_1 = \frac{1}{2}$. Then $|x-1| < \frac{1}{2} \Rightarrow 1 - |x| < |1-x| = |x-1| < \frac{1}{2}$

$\Rightarrow \frac{1}{2} < |x| \Rightarrow \underline{2 > \frac{1}{|x|}}$

also $\underline{|2x-1|} = |2(x-1)+1| \leq 2|x-1| + 1 \leq 2 \cdot \frac{1}{2} + 1 = \underline{2}$

Hence if $2 \cdot 2 |x-1| < \varepsilon$ we would have $|f(x) - f(1)| < \varepsilon$.

So let $\boxed{\delta_\varepsilon = \min\left\{\frac{1}{2}, \frac{\varepsilon}{4}\right\}}$ ... $\Rightarrow \lim_{x \to 1} f(x) = f(1)$.

Therefore $f(x)$ is <u>continuous</u> at $x = 1$.

Since $f(x) = 2x + \frac{1}{x}$, if x is near 0 and positive, $f(x)$ is large and positive, i.e. $\lim_{x \to 0+} f(x) = +\infty$. Similarly if $x \to 0-$, $f(x) \to -\infty$. If $x \to \pm\infty$, $\frac{1}{x} \to 0$, Hence $f(x) - 2x = \frac{1}{x} \to 0$ and $f(x)$ goes to $\pm\infty$ like $2x$.