Math 252 2008 Test I Solutions.

1. (a) $\lim_{x \to a+} f(x) = L$ means $\forall \varepsilon > 0 \ \exists \delta_\varepsilon > 0$ so.

$$a < x < a + \delta_\varepsilon \Rightarrow |f(x) - L| < \varepsilon.$$

(b) $\lim_{n \to \infty} a_n = \infty$ means $\forall M > 0 \ \exists N_M \in \mathbb{N}$ so

$$\forall n \geq N_M, \quad a_n > M \quad (\text{or } a_n \geq M).$$

(c) If $\overbrace{\exists r}^{a_n > 0 \ \forall n \in \mathbb{N} \ \&}$ with $0 < r < 1$ and $\frac{a_{n+1}}{a_n} \leq r$ then

$\sum_{n=1}^{\infty} a_n$ converges.

or If. $a_n > 0 \ \forall n \in \mathbb{N}$ and $\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = r < 1$ then

$\sum_{n=1}^{\infty} a_n$ converges.

If, in either case r exists but $r > 1$ then the series diverges.
If $r = 1$ the test fails.

(d) Let $f(n) = a_n$ with $a_n \geq 0$ and $a_{n+1} \leq a_n$ for all n. Then the series $\sum_{n=1}^{\infty} a_n$ converges if.

$\lim_{T \to \infty} \int_1^T f(x)\,dx < \infty$. The series diverges if

$\lim_{T \to \infty} \int_1^T f(x)\,dx = \infty$.

2) $\mathbb{R}$ satisfies the <u>least upper bound axiom</u> i.e. If $S \subset \mathbb{R}$ is bounded above, then S has a least upper bound i.e. an upper bound which is less than or equal to every other upper bound. Call this the <u>lub</u> of S.

2 cont.

Let $S = \{a_n \mid n \in \mathbb{N}\}$ be the set of values of the sequence and let $\varepsilon > 0$ be given. Since S is bounded above, it has a lub, α say. $\alpha = \text{lub}(S)$.

Then $\exists N_\varepsilon$ so $\alpha - \varepsilon < a_{N_\varepsilon} \leq \alpha$. But (a_n) is increasing

so $a_{N_\varepsilon} \leq a_{N_\varepsilon+1} \leq a_{N_\varepsilon+2} \leq \ldots$ and $\forall n \geq N_\varepsilon$, $a_{N_\varepsilon} \leq a_n$.

But α is the lub of S so is an upper bound for all of the sequence values. Thus

$$\alpha - \varepsilon < a_{N_\varepsilon} \leq a_n \leq \alpha < \alpha + \varepsilon \quad \forall n \geq N_\varepsilon.$$

Hence $\alpha - \varepsilon < a_n < \alpha + \varepsilon \quad \forall n \geq N_\varepsilon \implies |a_n - \alpha| < \varepsilon$

for these n. Therefore $\lim_{n \to \infty} a_n = \alpha$. ///

Let $a_n = 2 - \frac{3}{n}$. Then $a_{n+1} - a_n = 2 - \frac{3}{n+1} - \left(2 - \frac{3}{n}\right)$

$$= 3\left(\frac{1}{n} - \frac{1}{n+1}\right) = \frac{3}{n(n+1)} > 0$$

Therefore a_n is increasing. And $a_n < 2 \ \forall n \in \mathbb{N}$

Hence $\lim_{n \to \infty} a_n = \alpha = \text{lub}\left\{2 - \frac{3}{n} : n \in \mathbb{N}\right\}$ and this lub is 2 because $\forall \varepsilon > 0 \ \exists n$ with $2 - \varepsilon < 2 - \frac{3}{n} \leq 2$

namely, working back $2 - \varepsilon < 2 - \frac{3}{n}$

$$\iff \frac{3}{n} < \varepsilon \iff \frac{1}{n} < \frac{\varepsilon}{3}$$

& we can always find such an $n \in \mathbb{N}$. ///

3) Given $\varepsilon > 0$ $\exists N_1$ so $n \geq N_1 \Rightarrow |a_n - L| < \varepsilon/2$

$\exists N_2$ so $\forall n \geq N_2 \Rightarrow |b_n - M| < \varepsilon/2$

If. $N_\varepsilon = \max\{N_1, N_2\}$ and $n \geq N_\varepsilon$ then $|(a_n+b_n) - (L+M)| \leq |(a_n - L) + (b_n - M)|$

$\leq |a_n - L| + |b_n - M| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$

Hence $\lim\limits_{n\to\infty} a_n + b_n = L + M$.

4) (a) Given $\varepsilon > 0$, working back we want $\left| a_n - 5 \right| < \varepsilon$

$\Leftrightarrow \left| \frac{5n-2}{n+1} - 5 \right| < \varepsilon \Leftrightarrow \left| \frac{5n-2-5n-5}{n+1} \right| < \varepsilon \Leftrightarrow \frac{7}{n+1} < \varepsilon$

$\Leftrightarrow \frac{n+1}{7} > \frac{1}{\varepsilon} \Leftrightarrow n > 7\frac{1}{\varepsilon} - 1$. Hence let N_ε be any natural number with $N_\varepsilon > \frac{7}{\varepsilon} - 1$. If $n \geq N_\varepsilon \Rightarrow n > \frac{7}{\varepsilon} - 1$

and thus $|a_n - 5| < \varepsilon$, using the working back. Therefore $\lim\limits_{n\to\infty} a_n = 5$.

$$\lim_{n\to\infty} \frac{5n-2}{n+1} = \lim_{n\to\infty} \frac{5 - \frac{2}{n}}{1 + \frac{1}{n}} = \frac{\lim\limits_{n\to\infty}(5 - \frac{2}{n})}{\lim\limits_{n\to\infty}(1 + \frac{1}{n})} = \frac{5 - 2\lim\limits_{n\to\infty}\frac{1}{n}}{1 + \lim\limits_{n\to\infty}\frac{1}{n}} = \frac{5 - 2.0}{1+0}$$

$= \frac{5}{1} = 5$ where we have used the theorems

lt of a quot. is the quot of the limits, lt. of a sum is the sum of the limits, lt of a constant sequence is the constant value, and $1/n \to 0$.

(b) $a_n = \frac{1}{n+6}$ is decreasing because $a_n - a_{n+1} = \frac{1}{n+6} - \frac{1}{n+7}$ $= \frac{1}{(n+6)(n+7)} > 0$.

and $a_n > 0$. and $\lim\limits_{n\to\infty} n a_n = \lim\limits_{n\to\infty} \frac{n}{n+6} = 1 \neq 0$

therefore, by <u>Abel's test.</u> $\sum\limits_1^\infty a_n$ $\mathcal{D}$.

<u>Comparison</u> with $\sum\limits_1^\infty \frac{1}{n}$: $\frac{1}{n+6} > \frac{1/2}{n} \Leftrightarrow 2n > n+6 \Leftrightarrow n > 6$

since $\sum\limits_1^\infty \frac{1}{n}$ $\mathcal{D}$ so does $\sum\limits_1^\infty \frac{1/2}{n}$ and $\therefore$, by comparison, $\sum\limits_1^\infty \frac{1}{n+6} = \infty$.

5) Given $\varepsilon > 0$ let $\delta_1 = \frac{\varepsilon}{2\times 8}$. Then $|x-1| < \delta_1 \Rightarrow 8|x-1| < \frac{\varepsilon}{2}$

$\Rightarrow |8x - 8| < \frac{\varepsilon}{2}$ (1)

Working back: want $\left| \frac{2}{x} - \frac{2}{1} \right| < \frac{\varepsilon}{2}$

$\Leftrightarrow \left| \frac{2(x-1)}{x} \right| < \frac{\varepsilon}{2}$ so let. $\delta_2 = \frac{1}{2}$ so $|x-1| < \frac{1}{2}$

$\Rightarrow |1 - |x|| < \frac{1}{2}$

so if. $2 \cdot \frac{1}{2}|x-1| < \varepsilon/2$ we would attain the goal. $\Rightarrow \frac{1}{2} < |x|$

This is so if $|x-1| < \varepsilon/2$ so let.

$$\delta_\varepsilon = \min\left\{\frac{\varepsilon}{16}, \frac{1}{2}, \frac{\varepsilon}{2}\right\} = \min\left\{\frac{\varepsilon}{16}, \frac{1}{2}\right\}.$$

then if $|x-1| < \delta_\varepsilon$, we have

$$|f(x) - f(1)| = \left|\left(8x + \frac{2}{x} + 1\right) - \left(8 + \frac{2}{1} + 1\right)\right|$$

$$= \left|8(x-1) + \left(\frac{2}{x} - \frac{2}{1}\right)\right|$$

$$\leq |8x - 8| + \left|\frac{2}{x} - \frac{2}{1}\right|$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

and we have proved f is continuous at $x = 1$.

Near $x = 0+$, $8x$ is near 0, but $\frac{2}{x}$ is large and +ve.
& 1 near 1

Hence $\lim_{x\to 0+} f(x) = +\infty$. In contrast, if $x \to 0-$, $x<0$ so $8x$ is near 0, 1 near 1, but $\frac{2}{x}$ is large and −ve $\therefore \lim_{x\to 0-} f(x) = -\infty$.

When $x \to +\infty$ $8x+1 \to \infty$ and $\frac{2}{x} \to 0+$. Thus $f(x) \to \infty$.

note that $f(x) - 8x - 1 = \frac{2}{x} \to 0$ as $x \to \pm\infty$.