Elements of Analysis

Kevin A. Broughan
Department of Mathematics
University of Waikato
Private Bag 3105, Hamilton, New Zealand
kab@waikato.ac.nz
Notes by Melissa Sutjipto

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1 Inequalities and Absolute Value

Definition $a < b$ means b is to the right of a on $\mathbb{R}$. Hence if $a,\ b \in \mathbb{R}$ then $a < b,\ a = b$, or $b < a$ and exactly one of these is true.

Theorem 1 *If $a, b, c \in \mathbb{R}$ then*
a. $a < b \Rightarrow a + c < b + c$
b. $a < b$ *and* $c > 0 \Rightarrow ac < bc$
c. $a < b$ *and* $c < 0 \Rightarrow ac > bc$
d. $0 < a < b \Rightarrow 0 < \frac{1}{b} < \frac{1}{a}$
e. $a < b$ *and* $b < c \Rightarrow a < c$
f. $a < b$ *and* $a' < b' \Rightarrow a + a' < b + b'$

e.g. d. $-2 < 0 < 1$ $(a < 0 < b)$ but $\frac{1}{1} < \frac{1}{-2}$ is false

e.g. f.

$$a < b \Rightarrow a + a' < b + a'$$

by a.

$$a' < b' \Rightarrow b + a' < b + b'$$

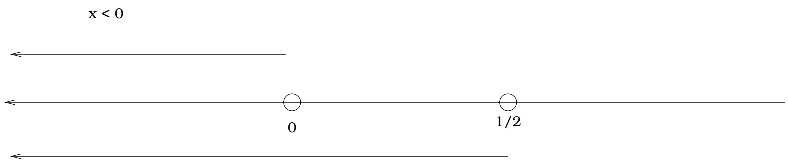
by a. thus by e. we get f.

e.g. d. $\frac{x}{y} = 0$ and $y \neq 0 \Rightarrow x = 0$
Hence $0 < a \Rightarrow 0 < \frac{1}{a}$ (otherwise $\frac{1}{a} < 0 \Rightarrow \frac{a}{a} < 0 \cdot a = 0 \Rightarrow 1 < 0$ false)

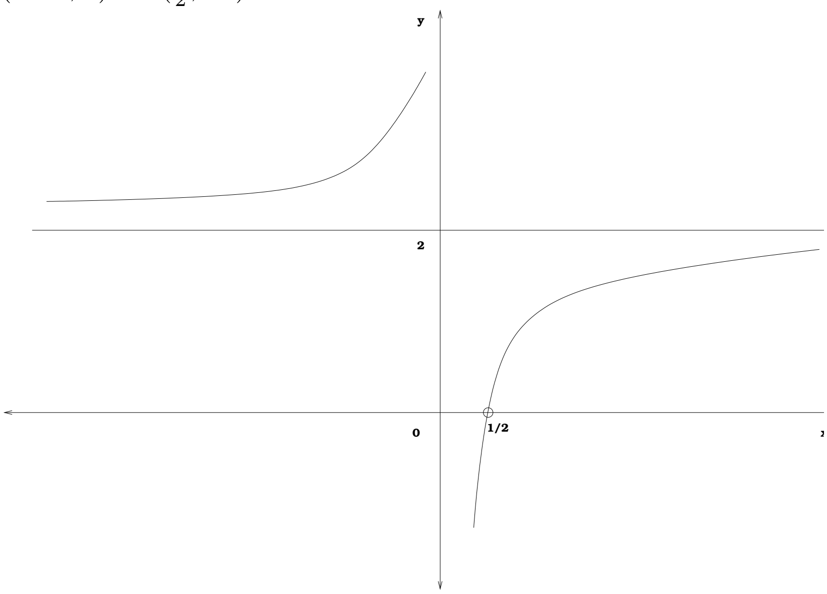
$$\therefore 0 < a < b \Rightarrow (\frac{1}{a})a < \frac{1}{a}b$$
$$\Rightarrow (\frac{1}{b})1 < (\frac{1}{b})(\frac{1}{a})b$$
$$\Rightarrow \frac{1}{b} < \frac{1}{a}$$

i.e. (**d.**)

e.g. Find $S = \{x \in \mathbb{R} : \frac{2x-1}{x} > 0\}$.
Then $\frac{2x-1}{x} > 0$ and $x > 0 \Rightarrow 2x - 1 > 0 \Leftrightarrow x > \frac{1}{2}$
and $\frac{2x-1}{x} > 0$ and $x < 0 \Rightarrow 2x - 1 < 0 \Leftrightarrow x < \frac{1}{2} \Leftrightarrow x < 0$



Hence $S = (-\infty, 0) \cup (\frac{1}{2}, \infty)$



OR

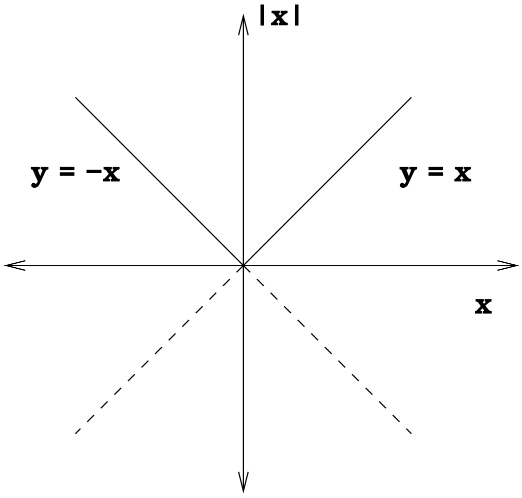
$$y = \frac{2x-1}{x} = \frac{2x}{x} - \frac{1}{x} = 2 - \frac{1}{x}$$

1.1 Absolute Value

e.g. $|-2| = 2$

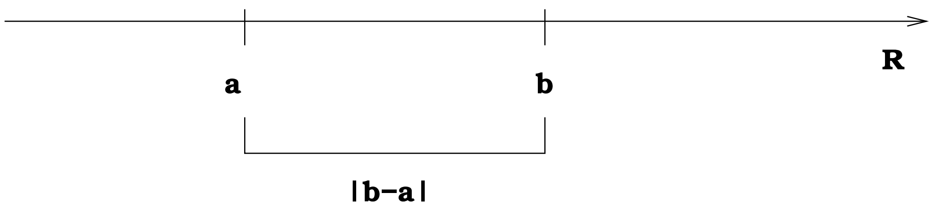
Definition

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$



e.g. $-2 < 0 \Rightarrow |-2| = -(-2) = 2$

e.g. $a, b \in \mathbb{R}$ $|a - b|$ is the numerical distance from a to b or b to a



Theorem 2
a. $|x| \geq 0$
b. $|x| \geq x$
c. $|x| \geq -x$
d. $|xy| = |x| \cdot |y|$
e. $|x + y| \leq |x| + |y|$ *($\triangle$ Law)*

f. $|x| - |y| \leq |x - y|$
g. $|x| = |-x|$

Note $a \leq b$ means $a < b$ or $a = b$. So $1 \leq 2$ and $-3 \leq -3$ are true.

Proof. **a, b, c, g** left to the student.
d. Case 1 $xy > 0 \Rightarrow x > 0$ and $y > 0$ or $x < 0$ and $y < 0$.
In the former instance, $|x| = x, |y| = y$ so $|x| \cdot |y| = xy$.
In the latter, $|x| = -x, |y| = -y$, so $|x| \cdot |y| = (-x)(-y) = xy$.
Hence $|x||y| = xy = |xy|$.

Case 2 $xy < 0$. Then either $x < 0$ and $0 < y$ or $0 < x$ and $y < 0$.
In the former, $|x| \cdot |y| = (-x)y = -xy$ and the same is true in the latter.
Hence $|x||y| = -xy = |xy|$.

e. Case 1 $x + y \geq 0 \Rightarrow |x + y| = x + y$.
By **b.**, $x \leq |x|, y \leq |y|$. By Thm 1 (f), $x + y \leq |x| + |y|$.
Hence $|x + y| = x + y \leq |x| + |y| \Rightarrow |x + y| \leq |x| + |y|$.

Case 2 $x + y < 0 \Rightarrow |x + y| = -(x + y) = -x - y$.
Use **c.** to get $-x \leq |x|$ and $-y \leq |y|$ and, again, $|x + y + \leq |x| + |y|$.

f. $|x| = |x - y + y| = |(x - y) + y| \leq |x - y| + |y|$.
By Thm 1 (a), $|x| - |y| \leq |x - y| + |y| - |y| \Rightarrow |x| - |y| \leq |x - y|$.

□

Theorem 3 *If* $\delta > 0$ *then* $|a - x| < \delta \Leftrightarrow a - \delta < x < a + \delta$

Proof. Assume $|a - x| < \delta$
$\Rightarrow a - x \leq |a - x| < \delta \Rightarrow a - \delta < x$ and
$\Rightarrow -(a - x) \leq |a - x| < \delta \Rightarrow x < a + \delta$

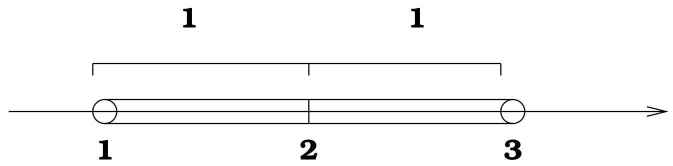
$$\therefore a - \delta < x < a + \delta.$$

Now assume $a - \delta < x < a + \delta$.
By Thm 1(a), $-\delta < x - a < \delta$.
If $x - a \geq 0 \Rightarrow |x - a| = x - a < \delta$ so $|x - a| < \delta$.
If $x - a < 0 \Rightarrow |x - a| = -(x - a) < \delta$ so $|x - a| < \delta$.

$\therefore$ since one of $x - a \geq 0$ and $x - a < 0$ must be true, we have $|x - a| < \delta$ in all cases.

$\square$

e.g. $|x - 2| < 1 \Leftrightarrow |2 - x| < 1 \Leftrightarrow 2 - 1 < x < 2 + 1 \Leftrightarrow 1 < x < 3$



e.g. $|x - 2| \geq 1 \Leftrightarrow$ not $1 < x < 3$
$\Leftrightarrow x \leq 1$ or $x \geq 3$

$$x \in (-\infty, 1] \cup [3, \infty)$$

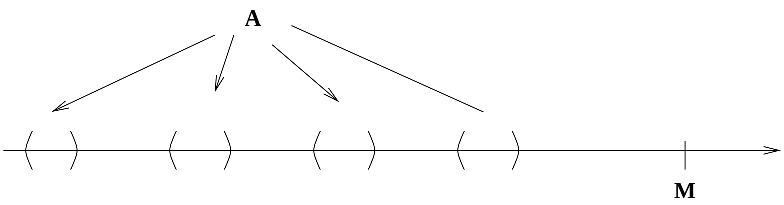
2 ℝ is complete

$\mathbb{Q} = \{\frac{m}{n} : m \in \mathbb{Z}, n \in \mathbb{N}\} \subset \mathbb{R}$
but $\mathbb{Q} \neq \mathbb{R}$.

e.g. $\sqrt{2} \notin \mathbb{Q}$.
Assume $\sqrt{2} \in \mathbb{Q}$(??).
So $\sqrt{2} = \frac{a}{b}$, $a, b \in \mathbb{N}$ and $(a, b) =$ gcd of a and $b = 1$.
Then $\sqrt{2}b = a \Rightarrow 2b^2 = a^2 \Rightarrow 2 \mid a^2$ (2 divides a)
$\Rightarrow a^2$ is even so a is even
(a odd $\Rightarrow a = 2c + 1 \Rightarrow a^2 = 4(c^2 + c) + 1$ odd also).
Hence $a = 2\alpha$
$\Rightarrow 2b^2 = 4\alpha^2$
$\Rightarrow b^2 = 2\alpha^2 \Rightarrow b^2$ is even $\Rightarrow b$ is even
$\Rightarrow 2 \mid a$ and $2 \mid b$ so $(a, b) \neq 1$, a contradiction (!!)
$\therefore \sqrt{2} \notin \mathbb{Q}$.

$\mathbb{Q}$ has many 'holes' but $\mathbb{R}$ has none. It is **complete**.

Definition $A \subset \mathbb{R}$ is **bounded above** if there exists an $M \in \mathbb{R}$ so for all $x \in$



A, $x \leq M$.

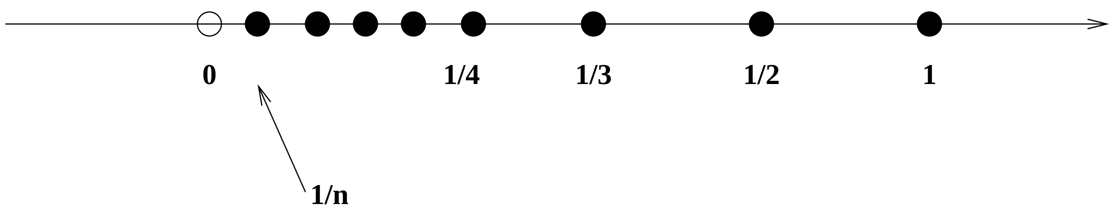
Notation: there exists $\exists$
for all $\forall$

$$\exists M \in \mathbb{R} \; so \; \forall x \in A, \; x \leq M$$

Say M is an **upper bound** for A.
Similarly, A is **bounded below** if $\exists m$ so $\forall x \in A, m \leq x$ and m is a **lower bound**. A is **bounded** if it has a lower and an upper bound (ub).

e.g. $\mathbb{N} = \mathbb{Z}_{>0} = \{1, 2, 3, ...\}$ has $m = 0$ or $m = -6$, or $m = 1$, so is bounded below.

e.g. $S = \{\frac{1}{n} : n \in \mathbb{N}\} = \{1, \frac{1}{2}, \frac{1}{3}, ...\}$ has $0 < x \leq 1$ so is bounded.



2.1 Completeness Axiom for $\mathbb{R}$

If $A \neq \emptyset$ is bounded above and then there is an $a \in \mathbb{R}$ so
(i) $\forall x \in A, \; x \leq a$,
(ii) if $\forall x \in A, x \leq b$ then $a \leq b$,
we call a the **least upper bound** or **supremum** of A, and write $a = lub(A) = sup(A)$.
(i) says a is an upper bound for A.

(ii) says a is less than every other upper bound for A, i.e. a is the least upper bound.

The **completeness axiom** states if $A \subset \mathbb{R}$ is nonempty and it has an upper bound, then it has a least upper bound.

Theorem 4 *If the lub of A exists then it is unique.*

Proof. If a_1 and a_2 are least upper bounds for A, then regarding a_1 as the lub and using a_2 as an ub $\Rightarrow a_1 \le a_2$. Regarding a_2 as the lub we get $a_2 \le a_1$. Thus $a_1 = a_2$, so the lub is unique. □

Theorem 5 *$A \neq \emptyset$ and given b satisfying*
(i) $\forall x \in A,\ x \le b$ (i.e. b is an upper bound)
(ii) $\forall \varepsilon > 0,\ \exists c \in A$ with $b - \varepsilon < c \le b$,
then $b =$ lub(A). Conversely, the lub of A satisfies (i) and (ii).

Proof. If a is an upper bound then $\forall \varepsilon > 0, \exists c \in A$ so

$$b - \varepsilon < c \le a \ \Rightarrow b < a + \varepsilon$$

If $a < b$ let $\varepsilon = b - a$ to derive $b < a + b - a = b$ (!!). Hence $b \le a$. Since a was an arbitrary upper bound, b must be the lub. □

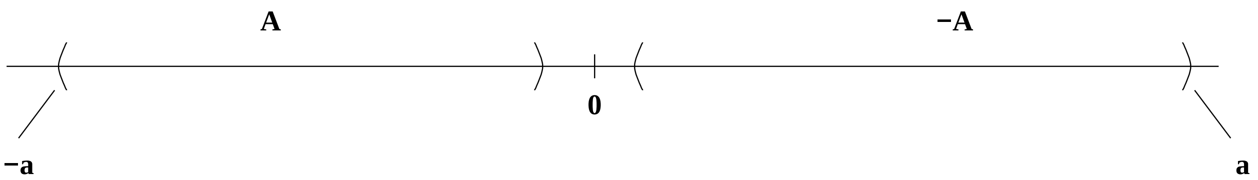
e.g. $\mathbb{Q}$ is **not** complete: $A = \{x \in \mathbb{Q} : x^2 < 2\}$

$$lub(A) = \sqrt{2} \notin \mathbb{Q}.$$

Similarly we can define the greatest lower bound (glb) or infimum (inf): $a = \inf A$ if (i) for all $x \in A$, $a \le x$, and (ii) for all $\varepsilon >$ there is a $c \in A$ so $a \le c < a + \varepsilon$.

Theorem 6 *If $A \subset \mathbb{R}$ has a lower bound then A has a glb.*

Proof. $b \le x\ \forall x \in A \Rightarrow -x \le -b$
$\Rightarrow -A = \{-x : x \in A\}$ is bounded above.
If $a = \sup(-A)$ then $-a = \inf(A)$ □



Theorem 7 (Archimedean Axiom) *Given* $b > 0$ *and* $a \geq 0$ *in* $\mathbb{R}$, $\exists n \in \mathbb{N}$ *with* $nb \geq a$.

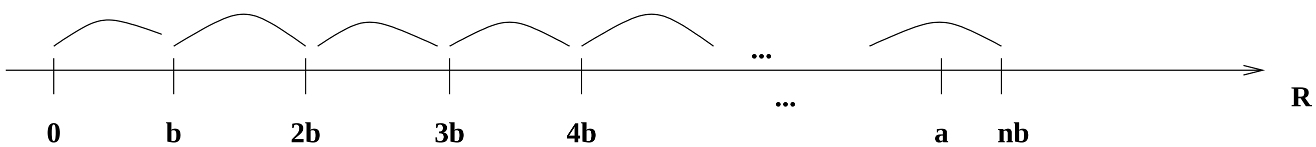
Proof. Suppose this is false. Then $\forall n \in \mathbb{N}, n < \frac{a}{b}, \Rightarrow \mathbb{N}$ is bounded above. Let $M = \text{lub}(\mathbb{N})$. Then, by Thm 4 with $\varepsilon = \frac{1}{2}, \exists n \in \mathbb{N}$

$$M - \frac{1}{2} < n < n + 1 \leq M$$

$$\Rightarrow 1 = (n+1) - n < M - (M - \frac{1}{2}) = \frac{1}{2}$$

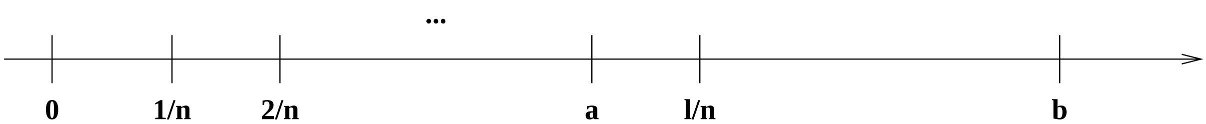
$\Rightarrow 1 < \frac{1}{2}$ (!!) false.
Thus $\exists n \in \mathbb{N}$ with $nb \leq a$. □

e.g. $\forall \varepsilon > 0, \exists n \in \mathbb{N}$ with $0 < \frac{1}{n} < \varepsilon$.
Let $b = \frac{\varepsilon}{2}$ and $a = 1 \Rightarrow \exists n$ s.t. $\frac{\varepsilon}{2} \cdot n \geq 1 \Rightarrow 0 < \frac{1}{n} \leq \frac{\varepsilon}{2} < \varepsilon$.



Theorem 8 *Let* $a < b \in \mathbb{R}$.
Then every open interval $(a, b) = \{x \in \mathbb{R} : a < x < b\}$ *contains an infinite number of rational numbers.*

Proof. It is sufficient to show it contains 1, i.e. $q_1 \in \mathbb{Q}$ since then the argument can be applied to (a, q_1) to get q_2 in $a < q_2 < q_1$ so $q_2 \neq q_1$ and then, by induction, $q_{n+1} \in (a, q_n)$ to get the infinite set $\{q_n\} \subset (a, b)$.



Let $\varepsilon = b - a > 0$ and choose $n \in \mathbb{N}$ so $0 < \frac{1}{n} < \varepsilon = b - a$. Let $a \geq 0$. By Thm 6 with $b \to \frac{1}{n}$ $\exists m \in \mathbb{N}$ so $m \cdot \frac{1}{n} > a$. Let l be the smallest such m. Then

$$a < \frac{l}{n}, \ \frac{l-1}{n} \leq a$$

.
If (??) $\frac{l}{n} \geq b$ then

$$\Rightarrow \frac{l-1}{n} \leq a < b \leq \frac{l}{n}$$
$$\Rightarrow \frac{1}{n} < b - a \leq \frac{l}{n} - \frac{l-1}{n} = \frac{1}{n}$$
$$\Rightarrow \frac{1}{n} < \frac{1}{n}(!!)$$

Hence $a < \frac{l}{n} < b$ so we can choose $q_1 = \frac{l}{n}$.
If $a < 0 < b$ apply the argument to $(0, b)$.
If $a < b \leq 0$ apply the argument to $(-b, -a)$ so $-\frac{l}{n}$ will be in (a, b). □

Definition $\text{lub}(S) = \infty \Leftrightarrow S$ has no u.b.
$\text{glb}(S) = -\infty \Leftrightarrow S$ has no l.b.
where ∞ satisfies $\forall a \in \mathbb{R}$:
$-\infty < a < \infty$
$\infty + a = \infty$
$-\infty + a = -\infty$
$\infty + \infty = \infty$
$-\infty - \infty = -\infty$
$a > 0 \Rightarrow \infty \cdot a = \infty$
But $\infty - \infty, \infty \cdot 0$ are undefined (since $\frac{a}{0}$).

Exercises on the lub:
(1) show $\{n + 1/n : n \in \mathbb{N}\}$ is not bounded above.
(2) show $\{(n+1)/n : n \in \mathbb{N}\}$ is bounded above.
(3) if $S = (0, 1]$ show that $lub\ S = 1$ and that $glb\ S = 0$.
(4) if $T = \{\frac{1}{n} : n \in \mathbb{N}\}$ show that $lub\ T = 1$ and $glb\ T = 0$.

3 Limit of a Sequence

A **sequence** is a function $f : \mathbb{N} \to \mathbb{R}$. Let $f(n) = a_n\ \forall n \in \mathbb{N}$. We use the notation (a_n) or $(a_n : n \in \mathbb{N})$ or $(a_1, a_2, ...)$ or $\{a_1, a_2, ...\}$ for the sequence f, but never the last form.

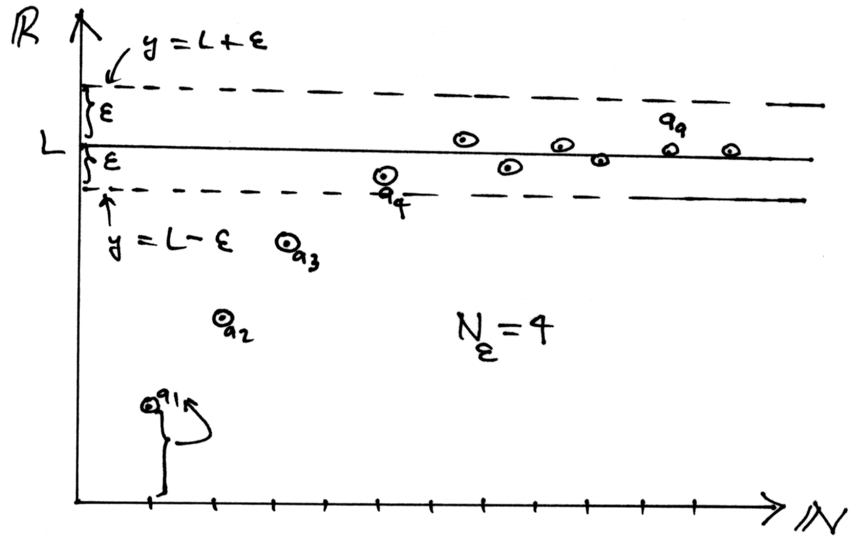


Figure 1: Fig1: Limit of a sequence

e.g.
(i) $(1, 1, 1, ...)$ $f(n) = a_n = 1 \, \forall n$, constant sequence.
(ii) $(1, \frac{1}{2}, \frac{1}{3}, ...)$ $f(n) = \frac{1}{n}$, decreasing sequence.
(iii) $(1, -1, 1, -1, ...)$ $f(n) = (-1)^{n+1}$, oscillating sequence.

Definition Let a_n be a sequence and $\alpha \in \mathbb{R}$. We say the sequence converges to α as a limit if the difference between a_n and α can be made **arbitrarily small** for all $n \in \mathbb{N}$ sufficiently large. If so, write

$$\lim_{x \to \infty} a_n = \alpha$$

.

Definition $\lim\limits_{n \to \infty} a_n = \alpha \Leftrightarrow \forall \varepsilon > 0, \exists N_\varepsilon \in \mathbb{N}$ s.t. $\forall n \geq N_\varepsilon$,

$$|a_n - \alpha| < \varepsilon$$

.

e.g. 1 $(1, 1, 1, ...) a_n = 1 \forall n \Rightarrow \lim_{n \to \infty} a_n = 1 = \alpha$.

Proof. Given $\varepsilon > 0$, let $N_\varepsilon = 1$.
Then if $n \geq N_\varepsilon = 1, |a_n - \alpha| = |1 - 1| = |0| = 0 < \varepsilon$.
Thus $\lim_{n\to\infty} a_n = 1$. □

e.g. 2 $a_n = \frac{1}{n} \forall n \in \mathbb{N}$. Then $\lim_{n\to\infty} a_n = 0 = \alpha$.
Proof. Given $\varepsilon > 0$, choose $N_\varepsilon \in \mathbb{N}$ so $0 < \frac{1}{N_\varepsilon} < \varepsilon$.
Then $\forall n \geq N_\varepsilon, |a_n - 0| = |\frac{1}{n}| = \frac{1}{n} \leq \frac{1}{N_\varepsilon} < \varepsilon$.
Hence $|a_n - 0| < \varepsilon$ so $\lim_{n\to\infty} \dfrac{1}{n} = 0$. □

e.g. 3 $a_n = \frac{n+1}{n}$ then $\lim_{n\to\infty} a_n = 1$ (Exercise).

e.g. 4 $a_n = (-1)^n$ then $\lim_{n\to\infty} a_n$ does **not** exist ($\nexists$).
Suppose the limit does exist and has value α. (??)
Let $\varepsilon = \frac{1}{2}$. $\exists N_{\frac{1}{2}}$ such that $|a_n - \alpha| < \frac{1}{2} \, \forall n \geq N_{\frac{1}{2}}$.
Take two consecutive values of$n \geq N_{\frac{1}{2}}$. Then

$$|1 - \alpha| < \frac{1}{2}$$

and

$$|-1 - \alpha| = |1 + \alpha| < \frac{1}{2}$$

$2 = |2| = |1 - \alpha + 1 + \alpha| \leq |1 - \alpha| + |1 + \alpha| < \frac{1}{2} + \frac{1}{2} = 1$
which is impossible!! Hence the limit does not exist.

Definition We say the sequence (a_n) **converges to** $+\infty \Leftrightarrow a_n$ can be made **as large as we wish** $\forall n \in \mathbb{N}$ sufficiently large, OR

$$\lim_{n\to\infty} a_n = \infty \Leftrightarrow \forall k > 0, \exists N_k \in \mathbb{N} \,\text{so}\, \forall n \geq N_k, a_n > k$$

.

e.g. $a_n = n^2, \lim_{n\to\infty} = \infty$. Given $k \geq 1$, let N_k be any whole number $> k$.
Then $\forall n \geq N_k, a_n = n^2 \geq {N_k}^2 > k^2 \geq k \Rightarrow a_n > k$.

Theorem 9 Uniqueness
Let $\lim_{n\to\infty} a_n = \alpha$ *and* $\lim_{n\to\infty} a_n = \alpha'$. *Then* $\alpha = \alpha'$.

That is to say, the limit, when it exists, is **unique**.

Proof. If $\alpha \neq \alpha'$ (??), let $\varepsilon = \frac{|\alpha-\alpha'|}{2} > 0$.
There is an N_ε such that $\forall n \geq N_\varepsilon,\ |a_n - \alpha| < \varepsilon$,
and an N'_ε s.t. $\forall n \geq N'_\varepsilon,\ \ |a_n - \alpha| < \varepsilon$.
Choose $n \geq max\ \{N_\varepsilon, N'_\varepsilon\}$. Then

$$|\alpha - \alpha'| = |\alpha - a_n + a_n - \alpha'|$$

$$= |(\alpha - a_n) + (a_n - \alpha')|$$

$$\leq |\alpha - a_n| + |a_n - \alpha'| \qquad \text{by}\triangle$$

$$< \frac{|\alpha - \alpha'|}{2} + \frac{|\alpha - \alpha'|}{2} = |\alpha - \alpha'|$$

$$\Rightarrow |\alpha - \alpha'| < |\alpha - \alpha'|\ (!!)$$

.
Hence $\alpha = \alpha'$. □

Theorem 10 Limit Theorem
Let $\lim_{n\to\infty} a_n = \alpha, \lim_{n\to\infty} b_n = \beta$ *and* $c \in \mathbb{R}$. *Then*
a. $\lim\limits_{n\to\infty} a_n + b_n = \alpha + \beta = \lim\limits_{n\to\infty} a_n + \lim\limits_{n\to\infty} b_n$
b. $\lim\limits_{n\to\infty} c \cdot a_n = c \cdot \alpha = c \lim\limits_{n\to\infty} a_n$
c. $\lim\limits_{n\to\infty} a_n \cdot b_n = \alpha \cdot \beta = (\lim\limits_{n\to\infty} a_n)(\lim\limits_{n\to\infty} b_n)$
d. *If* $\beta \neq 0$ *and* $b_n \neq 0\ \forall n, \lim\limits_{n\to\infty} \dfrac{1}{b_n} = \dfrac{1}{\beta}$

Proof.
a. Given $\varepsilon > 0$, because $a_n \to \alpha$ (i.e. $\lim\limits_{n\to\infty} a_n = \alpha$), $\exists N_1 \in \mathbb{N}$ so

$$\forall n \geq N_1,\ |a_n - \alpha| < \frac{\varepsilon}{2}$$

.

$$b_n \to \beta \Rightarrow \exists N_2 \,\text{so}\, \forall n \geq N_2,\ |b_n - \beta| < \frac{\varepsilon}{2}$$

.
Let $N_\varepsilon = max\{N_1,\ N_2\}$.

Then $n \geq N_\varepsilon \Rightarrow n \geq N_1$ and $n \geq N_2$ so $|a_n - \alpha| < \frac{\varepsilon}{2}$ and $|b_n - \beta| < \frac{\varepsilon}{2}$.
Hence $|(a_n + b_n) - (\alpha + \beta)| = |(a_n - \alpha) + (b_n - \beta)|$
$\leq |a_n - \alpha| + |b_n - \beta|$
$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$

Hence $\forall n \geq N_\varepsilon,\ |(a_n + b_n) - (\alpha + \beta)| < \varepsilon$.

$$\therefore \lim_{n\to\infty} a_n + b_n = \alpha + \beta$$

□

e.g. $\displaystyle\lim_{n\to\infty} \frac{n+1}{n} = \lim_{n\to\infty} 1 + \frac{1}{n} = \lim_{n\to\infty} 1 + \lim_{n\to\infty} \frac{1}{n} = 1 + 0 = 1$

b. If $c = 0$, the proof is an exercise $(0 \to 0)$.
If $c \neq 0$, replace ε by $\frac{\varepsilon}{|c|}$ in the definition for $a_n \to \alpha$.
Then $\exists N_\varepsilon$ s.t. $\forall n \geq N_\varepsilon,\ |a_n - \alpha| < \frac{\varepsilon}{|c|}$
$\Rightarrow |c||a_n - \alpha| < \varepsilon$
$\Rightarrow |c(a_n - \alpha)| < \varepsilon$
$\Rightarrow |c \cdot a_n - c \cdot \alpha| < \varepsilon$.

Hence $\lim_{n\to\infty} c \cdot a_n = c \cdot \alpha$.

c. Given $\varepsilon > 0, \exists N_1$ s.t. $\forall n \geq N_1,\ |a_n - \alpha| < 1$
$\Rightarrow |a_n| - |\alpha| \leq |a_n - \alpha| < 1$ by Thm 2 (f).
$\Rightarrow |a_n| < 1 + \alpha|$ (so (a_n) is bounded).

$$\exists N_2 \text{ so } \forall n \geq N_2,\ |a_n - \alpha| < \frac{\varepsilon}{2(|\beta| + 1)}$$

$$\exists N_3 \text{ so } \forall n \geq N_3,\ |b_n - \beta| < \frac{\varepsilon}{2(|\alpha| + 1)}$$

Let $N_\varepsilon = max\,\{N_1, N_2, N_3\}$ so $\forall n \geq N_\varepsilon$

$$|a_n b_n - \alpha\beta| = |a_n b_n - a_n\beta + a_n\beta - \alpha\beta|$$

$$\leq |a_n||b_n - \beta| + |\beta||a_n - \alpha|$$
$$< \frac{(1 + |\alpha|)\varepsilon}{2(|\alpha| + 1)} + \frac{|\beta|\varepsilon}{2(|\beta| + 1)}$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

Hence $a_n b_n \to \alpha\beta$.

d. Given $\varepsilon > 0, \exists N_1$ such that $\forall n \geq N_1,\ |b_n - \beta| < \frac{|\beta|}{2}$
$\Rightarrow |\beta| - |b_n| \leq |b_n - \beta| < \frac{|\beta|}{2}$
$\Rightarrow \frac{|\beta|}{2} < |b_n|$.

$\exists N_2$ so $\forall n \geq N_2,\ |b_n - \beta| < \frac{\varepsilon|\beta|^2}{2}$

Let $N_\varepsilon = max\{N_1, N_2\}$. Then $\forall n \geq N_\varepsilon$,

$$|\frac{1}{b_n} - \frac{1}{\beta}| = \frac{|b_n - \beta|}{|b_n||\beta|} < \frac{\varepsilon|\beta|^2}{2|\beta|\frac{|\beta|}{2}}$$

Hence $\frac{1}{b_n} \to \frac{1}{\beta} = \varepsilon$.

Corollary
$\frac{a_n}{b_n} \to \frac{\alpha}{\beta}$ Use Thm 10 (c), (d).

e.g.

$$\lim_{n\to\infty} \frac{n^2 + 3n + 1}{2n^2 + 3} = \lim_{n\to\infty} \frac{1 + \frac{3}{n} + \frac{1}{n^2}}{2 + \frac{3}{n^2}}$$

$$= \frac{\lim_{n\to\infty} 1 + \frac{3}{n} + \frac{1}{n^2}}{\lim_{n\to\infty} 2 + \frac{3}{n^2}}$$

$$= \frac{\lim_{n\to\infty} 1 + 3 \lim_{n\to\infty} \frac{1}{n} + (\lim_{n\to\infty} \frac{1}{n})^2}{2 \lim_{n\to\infty} 1 + 3(\lim_{n\to\infty} \frac{1}{n})^2}$$

$$= \frac{1 + 3 \cdot 0 + 0^2}{2 \cdot 1 + 3 \cdot 0^2} = \frac{1}{2}$$

4 Subsequences and Monotonic Sequences

Let $(n_1, n_2, n_3, ...)$ be a sequence in $\mathbb{N}$ with $n_1 < n_2 < n_3 < ...$. If (a_n) is a sequence, then so is $(a_{n_j} : j \in \mathbb{N})$ and we call it a **subsequence** of(a_n).
By induction we can prove that for all $i \geq 1$, $i \leq n_i$.

e.g. $n_j = 2j \; \forall j \in \mathbb{N}$. The subsequence is $(a_2, a_4, a_6, ...)$

e.g. $n_j = j^2 \; \forall j \in \mathbb{N}$. The subsequence is $(a_1, a_4, a_9, ...)$

Theorem 11 *If $a_n \to \alpha$ then any subsequence $a_{n_j} \to \alpha$.*

Proof. Let $\varepsilon > 0$ be given. Then $\exists N_\varepsilon$ so $\forall n \geq N_\varepsilon, |a_n - \alpha| < \varepsilon$.
But $\forall j, n_j \geq j$. Then $\forall j \geq N_\varepsilon, n_j \geq j \geq N_\varepsilon$
$\Rightarrow n_j \geq N_\varepsilon \Rightarrow |a_{n_j} - \alpha| < \varepsilon$.
Hence $a_{n_j} \to \alpha$. □

Note A non-convergent sequence can have several convergent subsequences $(0, 1, 2, 0, 1, 2, ...)$.

e.g. $\frac{1}{n} \to 0 \Rightarrow \frac{1}{n^2} \to 0, \frac{1}{n^3+n+1} \to 0$.

Completeness of $\mathbb{R}$ can be proved to be equivalent to "Every bounded sequence in $\mathbb{R}$ has a convergent subsequence".

Definition We say (a_n) is **increasing** if $a_n \leq a_{n+1} \; \forall n \in \mathbb{N}$.
We say (a_n) is **bounded above** if $a_n \leq M$ for some $M \in \mathbb{R}$ and all $n \in \mathbb{N}$.

Theorem 12 *Let (a_n) be increasing and bounded above. Then $\lim_{n\to\infty} a_n$ exists and has value $lub\{a_n : n \in \mathbb{N}\}$.*

Proof. Let $\alpha = lub\ \{a_n : n \in \mathbb{N}\}$ and let $\varepsilon > 0$ be given. Then $\alpha \in \mathbb{R}$ since $\exists M$ with $a_n \leq M \; \forall n \in \mathbb{N}$.
Since α is an upper bound, $\exists N_\varepsilon$ so $\alpha - \varepsilon < a_{N_\varepsilon} \leq \alpha$.

$$\forall n \geq N_\varepsilon, \alpha - \varepsilon < a_{N\varepsilon} \leq a_{N_\varepsilon+1} \leq ... \leq a_n \leq \alpha$$

since (a_n) is increasing.

$$\Rightarrow \forall n \geq N_\varepsilon,\ \alpha - \varepsilon < a_n \leq \alpha < \alpha + \varepsilon$$

$\Rightarrow \alpha - \varepsilon < a_n < \alpha + \varepsilon$
$\Rightarrow |a_n - \alpha| < \varepsilon$.
Hence $a_n \to \alpha$. □

e.g $a_1 = \sqrt{2}, a_2 = \sqrt{2+\sqrt{2}}, a_3 = \sqrt{2+\sqrt{2+\sqrt{2}}}, ..., \ a_{n+1} = \sqrt{2+a_n}$.
Then $a_1 < 2$, and if $a_n < 2, a_{n+1} < \sqrt{2+2} = 2$ also, so, by induction,

$$a_n < 2 \ \forall n \in \mathbb{N}$$

.
Hence $(a_n)$is bounded above.
Also, (a_n) is increasing: $a_1 < a_2$ and if $a_n < a_{n+1} ... \Rightarrow a_{n+1} < a_{n+2}$.
Hence $a_n \to \alpha$ for some α.
But ${a_{n+1}}^2 = 2 + a_n \Rightarrow \alpha^2 = 2 + \alpha \implies \alpha = 2$

Definition (a_n) is **decreasing** and bounded below...

Theorem 13 *If (a_n) is decreasing and bounded below, then $a_n \to \alpha$ for some $\alpha \in \mathbb{R}$.*

Proof. Let $b_n = -a_n$. Then $(b_n)$is increasing and bounded above so

$$\lim_{n\to\infty} b_n = \beta \ \exists \in \mathbb{R}$$

.
Thm 10 (b) $\Rightarrow \lim_{n\to\infty} -a_n = -\lim_{n\to\infty} a_n = \beta$
$\Rightarrow \lim_{n\to\infty} a_n = -\beta = \alpha$, say. □

Theorem 14 Sandwich Theorem
If $a_n \to \alpha$, $b_n \to \alpha$, and $a_n \leq c_n \leq b_n \, \forall n \in \mathbb{N}$, then $c_n \to \alpha$ also.

Proof. Let $\varepsilon > 0$ be given. Then, since $a_n \to \alpha, \exists N_1$ so

$$\forall n \geq N_1, \ |a_n - \alpha| < \varepsilon \Rightarrow \alpha - \varepsilon < a_n < \alpha + \varepsilon$$

$$\forall n \geq N_2, \ |b_n - \alpha| < \varepsilon \Rightarrow \alpha - \varepsilon < b_n < \alpha + \varepsilon$$

Let $N_\varepsilon = max\{N_1, N_2\}$. Then $n \geq N_\varepsilon \Rightarrow n \geq N_1$ and $n \geq N_2$.
So

$$\alpha - \varepsilon < a_n \leq c_n \leq b_n < \alpha + \varepsilon$$

$$\Rightarrow \alpha - \varepsilon < c_n < \alpha + \varepsilon \Rightarrow |c_n - \alpha| < \varepsilon.$$

□

e.g. $c_n = \frac{\sin n}{n^2+1}$
Now $|\sin\theta| \le 1 \Rightarrow -1 \le \sin n \le 1 \; \forall n \in \mathbb{N}$.
Hence $\frac{-1}{n^2+1} \le \frac{\sin n}{n^2+1} \le \frac{1}{n^2+1}$.
Now $a_n \to -0 = 0, b_n \to 0 \Rightarrow c_n \to 0$ by Thm 14.
Hence $\lim\limits_{n\to\infty} \dfrac{\sin n}{n^2+1} = 0$.

Some very useful limits

1. $|c| < 1 \Rightarrow c^n \to 0$
2. $1 < c \Rightarrow c^n \to \infty$
3. $\frac{\log n}{n} \to 0$
4. $n^{\frac{1}{n}} \to 0$
5. $0 < a \Rightarrow a^{\frac{1}{n}} \to 1$
6. $\forall b \in \mathbb{R}, \frac{b^n}{n!} \to 0$
7. $\left(1 + \frac{1}{n}\right)^n \to e$

e.g

$$a_n = \frac{\frac{2\ln n^2}{n} + n}{n^{\frac{1}{n}} + 2^{\frac{1}{n}} + 3n} = \frac{\frac{2\ln n^2}{n^2} + 1}{\frac{n^{\frac{1}{n}}}{n} + \frac{2^{\frac{1}{n}}}{n} + 3}$$
$$\to \frac{2\cdot 0 + 1}{1\cdot 0 + 1\cdot 0 + 3} = \frac{1}{3}$$

Proof.
1. Note that $|a_n - 0| = |a_n| = ||a_n|| = ||a_n| - 0|$.
Hence $a_n \to 0 \Leftrightarrow |a_n| \to 0$. Thus we can let $a_n = |c|^n$ or just assume $0 \le c$.
So let $0 \le c < 1$ and let $a_n = c^n$.
Then $a_{n+1} = c^{n+1} = c \cdot c^n = c \cdot a_n, \Rightarrow 0 < a_{n+1} < a_n$.
Therefore, the sequence (a_n) is monotonically decreasing and bounded below.
Therefore, $\lim\limits_{n\to\infty} a_n = L$ exists.
Now since $a_{n+1} = c \cdot a_n$,

$$\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} c \cdot a_n = c \lim_{n\to\infty} a_n$$
$$\Rightarrow \lim_{n\to\infty} a_{n+1} = c \lim_{n\to\infty} a_n$$

$$\Rightarrow L = cL$$
$$\Rightarrow L(1-c) = 0 \Rightarrow L = 0$$

Hence $\lim_{n\to\infty} c^n = 0$.

2. Let $1 < c$ and $a_n = c^n$. Then $a_{n+1} = c \cdot a_n > a_n$. Hence (a_n) is increasing. If it were bounded above, then $\lim_{n\to\infty} a_n = L$ would exist as a real number, and

$$\lim_{n\to\infty} a_{n+1} = c \lim_{n\to\infty} a_{n+1} \Rightarrow L = cL$$
$$\Rightarrow L(c-1) = 0 \Rightarrow L = 0,$$

which is impossible, since

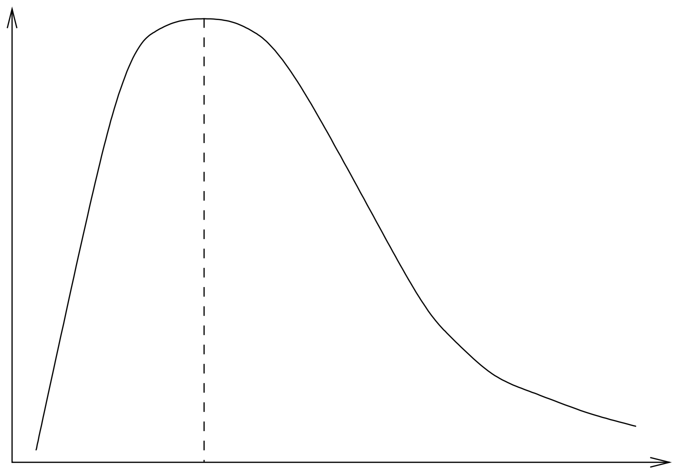
$$L = sup\{a_n\} \geq a_1 = c > 0.$$

Hence (a_n) is not bounded above. Therefore, by the theorem ((a_n) increasing and not bounded above implies $a_n \to \infty$)

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} c^n = \infty.$$

3. $a_n = \frac{\log n}{n}$.
Let $f(x) = \frac{\log x}{x}$

$$f : [1, \infty) \to \mathbb{R}$$
$$f'(x) = \frac{\frac{x}{x} - \log x}{x^2} = \frac{1 - \log x}{x^2}$$
$$\Rightarrow f'(x) < 0 \Leftrightarrow \log x > 1 \Leftrightarrow e^{logx} > e^1$$
$$\Leftrightarrow x > e.$$



Hence f is decreasing on (e, ∞).
Thus (a_n) is monotonically decreasing for $n \geq 3$.
Also, $0 \leq a_n \; \forall n \in \mathbb{N}$, so (a_n) is bounded below.
Hence $\lim_{n\to\infty} a_n = L$ exists in $\mathbb{R}$.
But then $\lim_{2n\to\infty} a_{2n} = L$ also.

$$a_{2n} = \frac{\log 2n}{2n} = \frac{\log 2}{2n} + \frac{\log n}{2n} = \frac{\log 2}{2} \cdot \frac{1}{n} + \frac{1}{2} \cdot \frac{\log n}{n}$$

Hence

$$\begin{aligned} L = \lim_{n\to\infty} a_{2n} &= \frac{\log 2}{2} \lim_{n\to\infty} \frac{1}{n} + \frac{1}{2} \lim_{n\to\infty} \frac{\log n}{n} \\ &= \frac{\log 2}{2} \cdot 0 + \frac{1}{2} L \\ &\Rightarrow \frac{1}{2} L = 0 \Rightarrow L = 0. \end{aligned}$$

Thus $\lim_{n\to\infty} \frac{\log n}{n} = 0$.

4. Use the theorem (proved later) that if f is a "nice" function (continuity at L is enough) and $a_n \to L$ then $f(a_n) \to f(L)$.

Now $\frac{\log n}{n} \to 0$ and $f(x) = e^x$ is nice.
Hence $e^{\frac{\log n}{n}} \to e^0$.

$$e^{\log n \frac{1}{n}} \to 1$$

$$n^{\frac{1}{n}} \to 1.$$

Hence $\lim_{n\to\infty} n^{\frac{1}{n}} = 1$.

5. $0 < a$.
$f(x) = a^x$ is nice and $\frac{1}{n} \to 0$.
Hence $f(\frac{1}{n}) \to f(0)$

$$a^{\frac{1}{n}} \to a^0.$$

$$\therefore \lim_{n\to\infty} a^{\frac{1}{n}} = 1.$$

6. Let $b \in \mathbb{R}$ and $a_n = \frac{b^n}{n!}$ and assume $b \geq 0$.
Then $a_{n+1} = \frac{b^{n+1}}{(n+1)!} = \frac{b}{n+1} \cdot \frac{b^n}{n!} = \frac{b}{n+1} \cdot a_n$.
Hence, from $N \in \mathbb{N}$ such that $b \leq N+1, (a_n)$ is decreasing and

$$a_n \geq 0 \; \forall n \in \mathbb{N}.$$

Hence the limit exists:

$$\lim_{n\to\infty} a_n = L.$$

Thus,

$$\lim_{n\to\infty} a_{n+1} = L = \lim_{n\to\infty} \frac{b}{n+1} \cdot \lim_{n\to\infty} a_n = 0 \cdot L \Rightarrow L = 0.$$

$$\therefore \lim_{n\to\infty} \frac{b^n}{n!} = 0.$$

Hence, by the remark in **1.**, $\frac{b^n}{n!} \to 0 \; \forall b \in \mathbb{R}$. □

5 Infinite Series of Positive Numbers I

e.g.
$1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + ... = 2$ "converges to 2"

$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + ... = \infty$ "diverges to ∞"

$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + ... = \infty$

If (a_n) is a sequence with $a_n \geq 0 \; \forall n \in \mathbb{N}$ form (s_n)

$$s_1 = a_1$$

$$s_2 = a_1 + a_2$$

$$s_3 = a_1 + a_2 + a_3$$

$$\vdots$$

$$s_{n+1} = s_n + a_{n+1}$$

Call (s_n) the **sequence of partial sums**. Then $s_{n+1} \geq s_n$ so

$$\lim_{n \to \infty} s_n = \begin{cases} s \in \mathbb{R} \; or \\ \infty \end{cases}$$

Indeed $\lim\limits_{n \to \infty} = s_n = lub\{s_n\} = lub\{a_1 + a_2 + ... + a_n\}$.

Write $\lim\limits_{n \to \infty} s_n = \sum\limits_{n=1}^{\infty} a_n$ and call the limit the **sum of the series**.
$= a_1 + a_2 + a_3 + ...$

Note (a_n) might start with $n = 0, n = -1$ and the theory is the same.

e.g. Geometric Series

$$r \geq 0, a_n = r^n, \; n \geq 0$$

$$\forall x \in \mathbb{R}, \; (1 - x^{n+1}) = (1 - x)(1 + x + x^2 + ... + x^n)$$

$$r \neq 1 \Rightarrow \frac{1 - r^{n+1}}{1 - r} = 1 + r + r^2 + ... + r^n$$

$$|r| < 1 \Rightarrow \sum_{n=0}^{\infty} = \lim_{n \to \infty} (1 + r + r^2 + ... + r^n) = \lim_{n \to \infty} \frac{1 - r^{n+1}}{1 - r} = \frac{1}{1 - r}$$

Use of series:
(1) Evaluating constants

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + ...$$

$$\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + ...$$

(2) Evaluating functions

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ...$$

$$\ln(1+x) = x - \frac{x}{2} + \frac{x}{3} - ...$$

Approach
1. Determine whether a finite sum exists
2. If so, evaluate the sum.

Theorem 15 *If* $\sum_{n=1}^{\infty} a_n$ *is convergent then* $a_n \to 0$.

Proof. Let $s_n = a_1 + ... + a_n$ and $\lim_{n\to\infty} s_n = s$.
Then $s_{n+1} = s_n + a_{n+1}$ by the limit theorem.

$$\lim_{n\to\infty} s_{n+1} = \lim_{n\to\infty} s_n + \lim_{n\to\infty} a_{n+1}$$

$$s = s + \lim_{n\to\infty} a_n \Rightarrow \lim_{n\to\infty} a_n = 0.$$

□

e.g. $\sum_{n=1}^{\infty} \frac{n^2}{n^2+1} = \infty$ since $a_n \to 1$, $\sum_{n=1}^{\infty} a_n$ diverges.

Note: Theorem 15 is a test for divergence.

Theorem 16

a. *If* $\sum_{n=1}^{\infty} a_n$ *is convergent then so is* $\sum_{n=N}^{\infty} a_n \; \forall N \geq 1$.

b. $\sum_{n=1}^{\infty} c \cdot a_n = c \sum_{n=1}^{\infty} a_n \; \forall c \geq 0$.

c. $\sum_{n=1}^{\infty} a_n + b_n = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$

Proof.

a. Recall that $a_n \geq 0$ for all n. Note $\sum_{n=1}^{\infty} a_n$ converges $\Leftrightarrow$ (s_n) is bounded above.

Let (t_n) be the partial sums for the second series for $n \geq N$.

Then $t_n = s_n - s_{N-1}$ where (s_n) are the partial sums for $\sum_{n=1}^{\infty} a_n$.

If $s_n \leq B$ $\forall n$ then $t_n \leq B - s_{N-1} \leq B$ $\forall n$ also.

Hence $\sum_{n=N}^{\infty} a_n$ converges also.

b. If $c = 0$ and $\sum_{n=1}^{\infty} a_n = \infty$ the result is false since $0 \cdot \infty$ is not defined.

If $c > 0$ and $\lim_{n\to\infty} s_n = s < \infty$ then the result follows from $\lim_{n\to\infty} cs_n = c \lim_{n\to\infty} s_n$.

c. Let $s_n = a_1 + a_2 + ... + a_n$

$t_n = b_1 + b_2 + ... + b_n$. Then

LHS $= \sum_{n=1}^{\infty} (a_n + b_n) = \lim_{n\to\infty} (a_1 + b_1 + ... + a_n + b_n)$

$= \lim_{n\to\infty} (a_1 + a_2 + ... + a_n + b_1 + b_2 + ... + b_n)$

$= \lim_{n\to\infty} (s_n + t_n)$

$= \lim_{n\to\infty} s_n + \lim_{n\to\infty} t_n = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n =$RHS. □

e.g.

$$2 + \frac{1}{2} + \frac{1}{3} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{2^3} + \frac{1}{3^3} + ...$$

$$= (1 + 1) + (\frac{1}{2} + \frac{1}{3}) + (\frac{1}{2^2} + \frac{1}{3^2}) + (\frac{1}{2^3} + \frac{1}{3^3}) + ...$$

$$= (1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + ...) + (1 + \frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + ...)$$

$= \frac{1}{1-\frac{1}{2}} + \frac{1}{1-\frac{1}{3}} = \frac{7}{2}$

Theorem 17 Comparison Test

a. *Let* $0 \le a_n \le b_n$ *and* $\sum_{n=1}^{\infty} b_n$ *converges then* $\sum_{n=1}^{\infty} a_n$ *converges also.*

b. *Let* $0 \le a_n \le b_n$ *and* $\sum_{n=1}^{\infty} a_n$ *diverges then* $\sum_{n=1}^{\infty} b_n$ *diverges also.*

Proof. Let

$$s_n = a_1 + ... + a_n$$

$$t_n = b_1 + ... + b_n$$

Then $0 \le s_n \le t_n \; \forall n$.
In **a.**, for some B, $t_n \le B \Rightarrow s_n \le B \Rightarrow$ result.
In **b.**, $s_n \le t_n$ and $s_n \to \infty \Rightarrow t_n \to \infty$ also. □

e.g. $\sum_{n=1}^{\infty} \frac{1}{n^n}$.
Now $n^n \ge 2^n \; \forall n \ge 2 \Rightarrow \frac{1}{n^n} \le \frac{1}{2^n}$.
Since $\sum_{n=0}^{\infty} \frac{1}{2^n} = 2 \Rightarrow \sum_{n=2}^{\infty} \frac{1}{2^n} = 2 - 1 - \frac{1}{2} = \frac{1}{2}$ Converges.
By Thm 17 (a), $\sum_{n=2}^{\infty} \frac{1}{n^n}$ converges, $\therefore \sum_{n=1}^{\infty} \frac{1}{n^n}$ converges.

e.g. $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ $a_n = \frac{1}{n(n+1)} \le \frac{1}{n^2} \; \forall n$.
Assuming $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges (below) $\Rightarrow \sum_{n=1}^{\infty} a_n$ converges also.

Note $a_n = \frac{1}{n} - \frac{1}{n+1}$ so

$$s_n = (1 - \frac{1}{2}) + (\frac{1}{2} - \frac{1}{3}) + ... + (\frac{1}{n} - \frac{1}{n+1})$$

$$= 1 - \frac{1}{n+1} \to 1$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1.$$

e.g. The comparison test is very powerful: If $f(n) \geq 0$,
$\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2 + f(n)}$ converges by the comparison test with $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$ for any such function $f(n)$.

Theorem 18 D'Alembert's Test I
Let $a_n > 0\ \forall n \in \mathbb{N}$.

a. *If* $\exists r$ *with* $0 < r < 1$ *and* $\frac{a_{n+1}}{a_n} \leq r\ \forall n$ *then* $\displaystyle\sum_{n=1}^{\infty} a_n$ *converges.*

b. *If* $\frac{a_{n+1}}{a_n} \geq 1\ \forall n$ *then* $\displaystyle\sum_{n=1}^{\infty} a_n = \infty$ *diverges.*

Proof.
a. $\frac{a_n}{a_1} = \frac{a_n}{a_{n-1}} \cdot \frac{a_{n-1}}{a_{n-2}} \ldots \frac{a_2}{a_1}$
$\leq r \cdot r \ldots r = r^{n-1} \quad \forall n \geq 1.$

Hence $s_N = \displaystyle\sum_1^N a_n \leq a_1 \sum_1^N r^{n-1} < \frac{a_1}{1-r} = B$.
$\Rightarrow s_N \to s \in \mathbb{R}$ and the series converges by the comparison test or directly since an increasing sequence bounded above converges.

b.

$$\frac{a_{n+1}}{a_n} \geq 1 \Rightarrow a_2 \geq a_1, \text{etc} \Rightarrow a_n \geq a_1 > 0$$

$\Rightarrow \displaystyle\lim_{n\to\infty} a_n \neq 0 \Rightarrow \sum_{n=1}^{\infty} a_n$ diverges. □

e.g.

$$a_n = \frac{1}{n!} \Rightarrow \frac{a_{n+1}}{a_n} = \frac{1}{n+1} \leq \frac{1}{2} < 1$$

$\Rightarrow \displaystyle\sum_{n=1}^{\infty} \frac{1}{n!}$ converges.

Theorem 19 D'Alembert's Test II
Let $a_n > 0 \; \forall n$. Then

a. $\displaystyle\lim_{n\to\infty} \frac{a_{n+1}}{a_n} < 1 \Rightarrow \sum_{n=1}^{\infty} a_n$ *converges.*

b. $\displaystyle\lim_{n\to\infty} \frac{a_{n+1}}{a_n} > 1 \Rightarrow \sum_{n=1}^{\infty} a_n$ *diverges.*

c. $\displaystyle\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1$ *then both convergence and divergence are possible.*

Proof.
a. Let $\frac{a_{n+1}}{a_n} \to r < 1$ and s satisfy $r < s < 1$ and $\varepsilon = s - r$.
$\exists N_\varepsilon$ so $\forall n \geq N_\varepsilon \; |\frac{a_{n+1}}{a_n} - r| < \varepsilon = s - r$.

$$\Rightarrow r - s < \frac{a_{n+1}}{a_n} - r < s - r \Rightarrow \frac{a_{n+1}}{a_n} < s < 1.$$

$\therefore \displaystyle\sum_{N_\varepsilon}^{\infty} a_n$ converges $\Rightarrow \displaystyle\sum_{n=1}^{\infty} a_n$ converges.

b. If $\frac{a_{n+1}}{a_n} \to r > 1$, let $\varepsilon = r - 1$. $\exists N_\varepsilon$ so $1 - r < \frac{a_{n+1}}{a_n} - r < r - 1 \Rightarrow \frac{a_{n+1}}{a_n} > 1$
$\Rightarrow \displaystyle\sum_{N_\varepsilon}^{\infty} a_n$ diverges $\Rightarrow \displaystyle\sum_{n=1}^{\infty}$ diverges also. □

e.g.
i. $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$ where $x \in (0, \infty)$ is a parameter.
$\frac{a_{n+1}}{a_n} = \frac{x}{n+1} \to 0 \Rightarrow \displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges.

ii. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n} : \frac{a_{n+1}}{a_n} = \frac{n}{n+1} \to 1$, so the test fails (later we show $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n} = \infty$).

iii. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} : \frac{a_{n+1}}{a_n} = \frac{n^2}{(n+1)^2} \to 1$ and it fails again.

Theorem 20 Cauchy's Test
a. *If* $a_n^{\frac{1}{n}} \to r < 1 \Rightarrow$ *convergent.*
b. *If* $a_n^{\frac{1}{n}} \to r > 1 \Rightarrow$ *divergent.*
c. *If* $a_n^{\frac{1}{n}} \to 1$ *the test is inconclusive.*

Proof.
a. Let $r < s < 1$ and $\varepsilon = s - r$. Then $\exists N_\varepsilon$ so $\forall n \geq N_\varepsilon$, $r - s < {a_n}^{\frac{1}{n}} - r < s - r$ $\Rightarrow {a_n}^{\frac{1}{n}} < s \Rightarrow a_n < s^n$ so, by comparison with a geometric series, $\sum_{n=1}^{\infty} a_n$ converges.

b. If ${a_n}^{\frac{1}{n}} \to r > 1 \Rightarrow \forall n \geq N_\varepsilon, {a_n}^{\frac{1}{n}} \geq 1 \Rightarrow a_n \geq 1^n = 1$
$\Rightarrow a_n$ does not approach 0. Hence the series diverges. □

e.g. $r \in (0, \infty), a_n = (\frac{nr}{n+1})^n$

$$\Rightarrow {a_n}^{\frac{1}{n}} = \frac{nr}{n+1} \to r.$$

Hence $0 < r < 1 \Rightarrow$ convergent and $1 < r \Rightarrow$ divergent.
If $r = 1$,

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} (\frac{n}{n+1})^n$$

$$= \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{n}\right)^n} = \frac{1}{\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n}$$

$= \frac{1}{e} \neq 0 \Rightarrow$ divergent.

Theorem 21 Abel's Test (for divergence)
If $a_n \geq a_{n+1}\ \forall n$ *and* $\sum a_n$ *is convergent, then* $\Rightarrow na_n \to 0$.

Proof. Let $s_n = \sum_1^n a_i = a_1 + ... + a_n$

$$\Rightarrow s_{2n} - s_n = \sum_{i=n+1}^{2n} a_i.$$

Let $s = \lim_{n\to\infty} s_n = \lim_{n\to\infty} s_{2n} \Rightarrow \sum_{i=n+1}^{2n} a_i = 0$.

Hence, given $\varepsilon > 0$, $\exists N_\varepsilon$ so $\forall n \geq N_\varepsilon$, $\sum_{i=n+1}^{2n} a_i < \frac{\varepsilon}{4}$.

Since (a_n) is decreasing,

$$na_{2n} \leq \sum_{i=n+1}^{2n} a_i < \frac{\varepsilon}{4} \Rightarrow 2na_{2n} < \frac{\varepsilon}{2} \;\forall n \geq N_\varepsilon.$$

Finally, $0 \leq (2n+1)a_{2n+1} = \frac{2n+1}{2n} \cdot 2na_{2n+1} \leq \frac{2n+1}{2n} \cdot 2na_{2n} \leq \frac{2n+1}{2n} \cdot \frac{\varepsilon}{2} < \varepsilon$.

$$\therefore \forall m \geq 2N_\varepsilon, ma_m < \varepsilon \Rightarrow na_n \to 0.$$

□

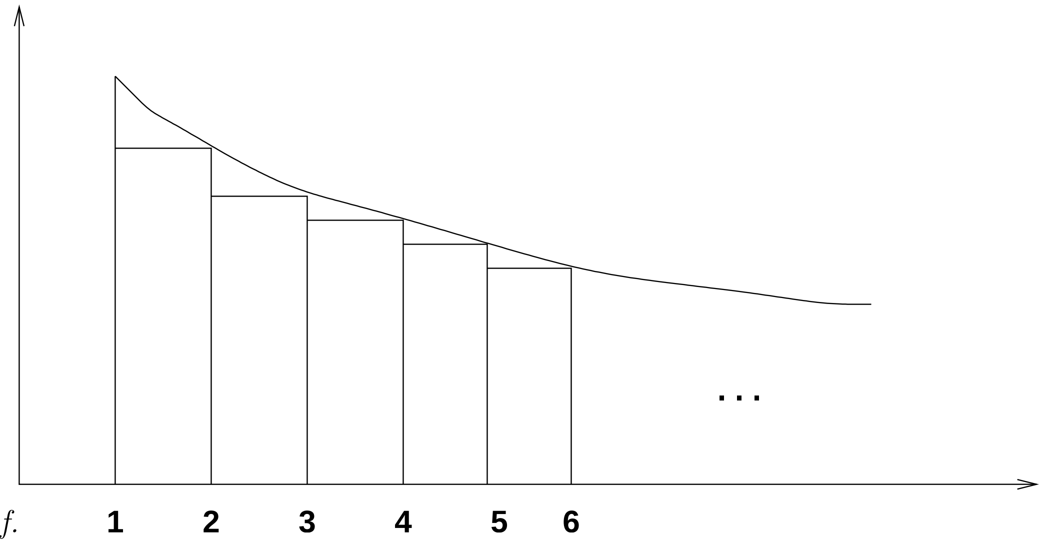
e.g. $\sum_{n=1}^{\infty} \frac{1}{n}$ has $na_n \to 1 \Rightarrow$ divergence.

e.g. $\sum_{n=3}^{\infty} \frac{1}{n \log n}$ has $na_n = \frac{1}{\log n} \to 0$ but it diverges, i.e. the test cannot be used: it can only be used when $na_n \not\to 0$.

6 Series of Positive Terms II

Theorem 22 Integral Test *Let $f : [1, \infty) \to \mathbb{R}$ be decreasing, positive, and continuous (i.e. $f(x) \geq 0\; \forall x \geq 1$ and $x < y \Rightarrow f(x) \leq f(y)$). Then*

$$\lim_{t\to\infty} \int_1^t f(x)\, dx < \infty \Leftrightarrow \sum_{n=1}^{\infty} f(n) < \infty.$$

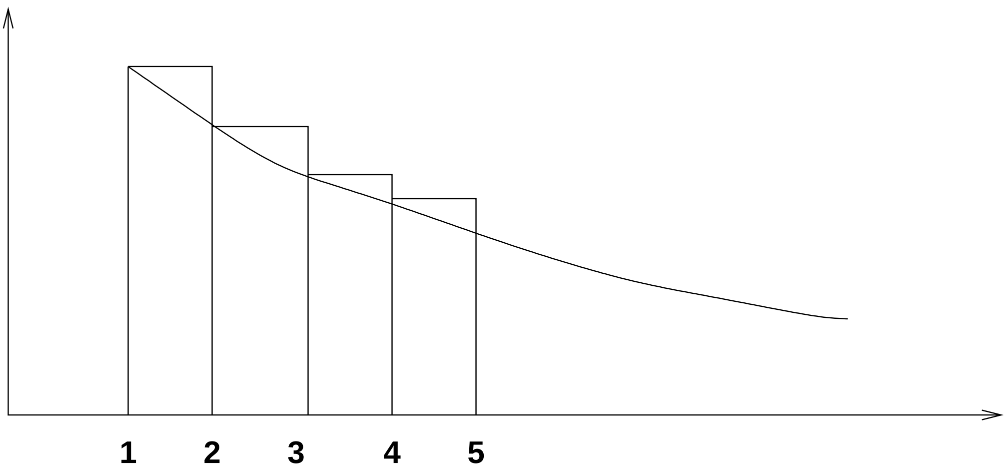


Proof.

Let $\lim_{t\to\infty} \int_1^t f(x)\,dx = B < \infty$.

Then $s_n = \sum_{j=2}^{n} f(j) \leq \int_1^n f(x)\,dx \leq B \;\forall n \in \mathbb{N} \setminus \{1\}$.

$$\Rightarrow \sum_1^\infty f(n) < \infty.$$



Let $\sum_{n=1}^{\infty} f(n) = s < \infty$

$$s \geq s_n = f(1) + f(2) + ...f(n)$$

$\geq \int_1^n f(x)\, dx \; \forall n \in \mathbb{N}$.

Hence $\lim_{t\to\infty} \int_1^t f(x)\, dx < \infty$. □

e.g. $a_n = \frac{1}{n} \Rightarrow f(x) = \frac{1}{x}$
and $\lim_{t\to\infty} \int_1^t \frac{1}{x}\, dx = \lim_{t\to\infty} \ln t = \infty \Rightarrow \sum_{n=1}^{\infty} \frac{1}{n} = \infty$.

e.g. $a_n = \frac{1}{n \ln n}$ $f(x) = \frac{1}{x \ln x}$

$$\lim_{t\to\infty} \int_2^t \frac{1}{x \ln x}\, dx = \lim_{t\to\infty} \ln \ln x \,|_2^t = \infty.$$

$\Rightarrow \sum_{n=2}^{\infty} a_n$ diverges.

e.g. Riemann Zeta Function

$$a_n = \frac{1}{n^s}, s \in (0, \infty) \text{ a parameter, } s > 0.$$

$$f(x) = \frac{1}{x^s}$$

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \text{ the zeta function.}$$

$$s = 1,\ \zeta(1) = \infty.$$

$$s \neq 1,$$

$$\lim_{t\to\infty}\int_1^t \frac{dx}{x^s} = \lim_{t\to\infty}\frac{x^{1-s}}{1-s}\Big|_1^t = \frac{1}{1-s}(\lim_{t\to\infty} t^{1-s}-1)$$
$$= \begin{cases} \frac{1}{s-1}, s>1 \\ \infty,\ 0<s<1 \end{cases}$$

So $\zeta(s) < \infty \Leftrightarrow s > 1$.

Note D'Alembert's test fails $\forall s$ since $\frac{a_{n+1}}{a_n} = \frac{n}{n+1}^s \to 1$.
Cauchy's test fails since $a_n^{\frac{1}{n}} = n^{\frac{1}{n}^{-s}} \to 1^{-s} = 1$ also.

Theorem 23 (Sufficient condition for Convergence)
If $\exists s > 1$ so $n^s a_n \to 0$ then $\sum_{n=1}^{\infty} a_n$ converges.

Proof. $n^s a_n \to 0 \Rightarrow \exists N_1$ so $n^s a_n < 1\ \forall n \geq N_1 \Rightarrow a_n \leq \frac{1}{n^s}$ and the comparison test shows convergence. □

e.g. Euler's Constant

$$\lim_{n\to\infty} 1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{n} - \ln(n+1) = \gamma = 0.5...$$

Let
$$b_n = \int_0^1 \frac{x}{n(n+x)}\,dx,\ c_n = \frac{1}{2n^2} = \int_0^1 \frac{x}{n^2}\,dx$$

Then
$$b_n = \int_0^1 \frac{1}{n} - \frac{1}{n+x}dx = \frac{x}{n} - \ln(n+x)|_0^1 = \frac{1}{n} - \ln(\frac{n+1}{n})$$

Because for $0 \leq x \leq 1$, $\frac{1}{n(n+x)} \leq \frac{1}{n^2}$ we get
$$b_n = \int_0^1 \frac{x}{n(n+x)}\,dx \leq \int_0^1 \frac{x}{n^2}dx = c_n$$
.

Also, $\sum_1^{\infty} c_n = \frac{1}{2}\zeta(2) < \infty$

$\therefore \sum_1^{\infty} b_n$ converges, so $\lim_{n\to\infty} b_1 + ... + b_n \exists$, call it γ.

Finally,

$$b_1 + ... + b_n = 1 + \frac{1}{2} + ...\frac{1}{n} - \ln\frac{2}{1} - \ln\frac{3}{2} - ... - \ln(\frac{n+1}{n})$$

$$= 1 + \frac{1}{2} + ...\frac{1}{n} - \ln(n+1).$$

7 Conditional and Absolute Convergence

Now let $a_n \in \mathbb{R} = (-\infty, \infty)$ so $a_n < 0$ is possible.
Again $s_n = a_1 + ... + a_n$ and $\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} s_n = s$ and, as before, $-\infty < s < \infty \Rightarrow a_n \to 0$.
If $a_n = (-1)^{n+1} b_n$ where $b_n \geq 0\ \forall n$ we say $\sum_{n=1}^{\infty} a_n$ is **alternating**.

e.g. $\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + ... \quad b_n = \frac{1}{n}$

Theorem 24 Leibniz
If $\sum_{n=1}^{\infty} a_n$ *is alternating and* $b_n \to 0$ *and* $b_n \geq b_{n+1} \geq 0\ \forall n \in \mathbb{N}$ *then* $\sum_{n=1}^{\infty} a_n$ *converges.*

Proof. Let $s_n = \sum_{j=1}^{n} (-1)^{j+1} b_j$.

$$\Rightarrow s_{2n+1} - s_{2n-1} = b_{2n+1} - b_{2n} \leq 0$$

and

$$s_{2n} - s_{2n-2} = -b_{2n} + b_{2n-1} \geq 0.$$

Thus the odd subsequence of partial sums $(s_1, s_3, s_5, ...)$ is decreasing and the even subsequence of partial sums $(s_2, s_4, ...)$ is increasing.
Thus

$$\lim_{n\to\infty} s_{2n+1} = r \text{ or } -\infty$$

$$\lim_{n\to\infty} s_{2n} = s \text{ or } +\infty.$$

But $s_{2n+1} - s_{2n} = a_{2n+1} \to 0$. Then both limits are finite and $r = s$.

$\therefore \lim_{n\to\infty} s_n = s$ also and $\sum_{n=1}^{\infty} (-1)^{n+1} b_n$ converges. □

e.g. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^{\frac{1}{100}}}$ is convergent.

Definition a series $\sum_{n=1}^{\infty} a_n$ is **absolutely convergent** if $\sum_{n=1}^{\infty} |a_n|$ is convergent.

e.g. $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$

Definition $\sum_{n=1}^{\infty} a_n$ is **conditionally convergent** if it is convergent but **not** absolutely convergent.

e.g. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ is conditionally convergent.

Theorem 25 *If* $\sum_{n=1}^{\infty} a_n$ *is absolutely convergent, then it is convergent.*

Proof. Given $\sum_{n=1}^{\infty} |a_n|$ is convergent, let $b_n = |a_n| + a_n$.

Now $0 \le b_n \le 2|a_n|$ so $\sum_{n=1}^{\infty} b_n$ is a series of positive terms which converges by comparison.

But then

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (b_n - |a_n|) = \sum_{1}^{\infty} b_n - \sum_{n=1}^{\infty} |a_n|$$

$$\Rightarrow \sum_{n=1}^{\infty} a_n \text{ converges.}$$

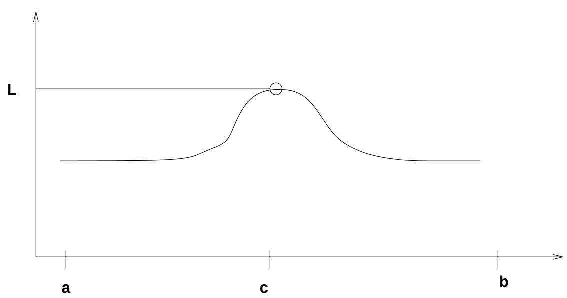
□

e.g.
i. $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ is conditionally convergent.
ii. $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n!}$ is absolutely convergent $\forall x \in \mathbb{R}$ since

$$a_n = \frac{x^n}{n!} \Rightarrow |a_n| = \frac{|x|^n}{n!}, \frac{|a_{n+1}|}{|a_n|} = \frac{|x|}{n+1} \to 0.$$

iii. $\displaystyle\sum_{n=1}^{\infty}(-1)^n$ neither converges nor diverges to $\pm\infty$ since $a_n \not\to 0$.
iv. $\displaystyle\sum_{n=1}^{\infty}(-n)(-1)^{n+1} = 1 - 2 + 3 - 4 + 5...$ oscillates infinitely.

8 Limits of Functions



Let $f : (a, b) \setminus \{c\} \to \mathbb{R}$ be a function. Then $\lim_{n\to c} f(x) = L$ means $\forall\varepsilon > 0\ \exists\delta_\varepsilon > 0$ such that if $0 < |x - c| < \delta_\varepsilon$ then $|f(x) - L| < \varepsilon$.

Note
1. This means $f(x)$ becomes arbitrarily close to L provided we choose x sufficiently close to c.
2. We never have $x = c$ since $|x - c| > 0 \Leftrightarrow x \neq c$.
3. $f(c)$ may very well be defined.

e.g. $f(x) = 2x,\ x \in \mathbb{R}\ \lim_{x \to 7} f(x) = 14.$
Let $\varepsilon > 0$ be given. Working back, want $|f(x) - 14| < \varepsilon$

$$\Leftrightarrow |2x - 14| < \varepsilon$$

$$\Leftrightarrow |x - 7| < \frac{\varepsilon}{2}$$

So let $\delta_\varepsilon = \frac{\varepsilon}{2}$. Then $0 < |x - 7| < \delta_\varepsilon \Rightarrow |x - 7| < \frac{\varepsilon}{2}$

$$\Rightarrow |f(x) - 14| < \varepsilon$$

$$\therefore \lim_{x \to 7} 2x = 14.$$

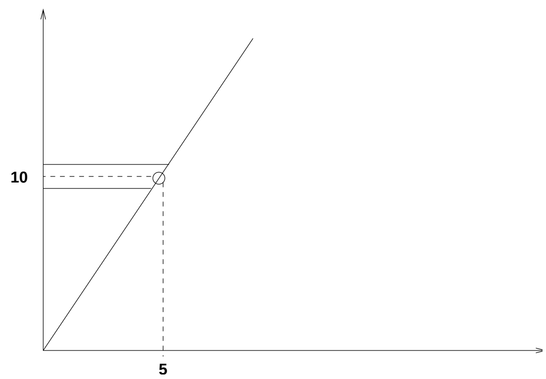
Note Since $14 = f(7)$ we say f is **continuous** at $x = 7$.

Definition f is **continuous** at $x = c \Leftrightarrow \lim_{x \to c} f(x) = f(c)$.

Note To be continuous at $x = c$, c must be in the domain of f.

e.g.

$$f(x) = \frac{2x(x-5)}{x-5} = y \quad f : \mathbb{R} \setminus \{5\} \to \mathbb{R}.$$



Given $\varepsilon > 0$, let $\delta_\varepsilon = \frac{\varepsilon}{2}$. Then $0 < |x - 5| < \frac{\varepsilon}{2}$

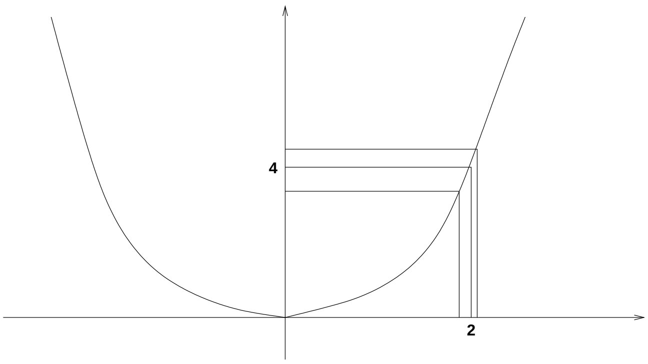
$$\Rightarrow 0 < |2x - 10| < \varepsilon \text{ and } x \neq 5.$$

$$\Rightarrow |\frac{2x(x-5)}{x-5} - 10| < \varepsilon$$

$$\Rightarrow |f(x) - 10| < \varepsilon.$$

Hence $\lim_{x \to 5} f(x) = 10$
But f is **not** continuous at $x = 5$.

e.g. $f(x) = x^2$ then $\lim_{x \to 2} f(x) = 4$.



Given $\varepsilon > 0$, want $|x^2 - 4| < \varepsilon$

$$\Leftrightarrow |x + 2||x - 2| < \varepsilon.$$

Let $\delta_1 = 1$. Then $0 < |x - 2| < \delta_1 \Rightarrow |x - 2| < \delta_1 = 1$

$$\Rightarrow |x + 2| = |x - 2 + 4| \leq |x - 2| + 4 < 5.$$

So if $5|x - 2| < \varepsilon \Rightarrow |x + 2||x - 2| < \varepsilon \Rightarrow |x^2 - 4| < \varepsilon$.
Let $\delta_\varepsilon = min\{1, \frac{\varepsilon}{5}\}$.

Note $f(x) = x^2$ is defined on $\mathbb{R}$ and $f(2) = 4$ so f is continuous at $x = 2$.
This, of course, is true at every $c \in \mathbb{R}$.

e.g. $\forall c \in \mathbb{R}$, $\lim_{x \to c} x^2 = c^2$.

Proof. Given $\varepsilon > 0$, want $|x^2 - c^2| < \varepsilon$
$\Leftrightarrow |x + c||x - c| < \varepsilon$.
Let $\delta_1 = 1$ so $0 < |x - c| < 1 \Rightarrow |x + c| = |x - c + 2c| \leq |x + c| + 2|c| < 1 + 2|c|$.
So if $(1 + 2|c|)|x - c| < \varepsilon$ we would have

$|x+c||x-c| < \varepsilon$ and we would be done.
So let $\delta_\varepsilon = min\{1, \frac{\varepsilon}{1+2|c|}\}$ □

e.g.

$$f : (0, \infty) \to \mathbb{R}.$$
$$x \mapsto \sqrt{x}$$

Then $\forall c > 0, \lim_{x\to c} \sqrt{x} = \sqrt{c}$.

Proof. Let $\varepsilon > 0$ be given. Working back: we want

$$|f(x) - \sqrt{c}| < \varepsilon \Leftrightarrow |\sqrt{x} - \sqrt{c}| < \varepsilon$$
$$\Leftrightarrow |\frac{(\sqrt{x}-\sqrt{c})(\sqrt{x}+\sqrt{c})}{\sqrt{x}+\sqrt{c}}| < \varepsilon$$
$$\Leftrightarrow \frac{|x-c|}{\sqrt{x}+\sqrt{c}} < \varepsilon$$

But $\sqrt{c} > 0$ and $\sqrt{x} + \sqrt{c} > \sqrt{c}$.
So if we had $\frac{|x-c|}{\sqrt{c}} < \varepsilon$ we would be done.
Hence $\delta_\varepsilon = \sqrt{c} \cdot \varepsilon$ □

Theorem 26 *If $f : (a,b) \setminus \{c\} \to \mathbb{R}$ and $\lim_{x\to c} f(x)$ exists, then it is unique.*

Proof. Let $\lim_{x\to c} f(x) = L_1$ and $\lim_{x\to c} f(x) = L_2$ and then given $\varepsilon > 0$,

$$\exists \delta_1 \text{ so } 0 < |x-c| < \delta_1 \Rightarrow |f(x) - L_1| < \varepsilon,$$
$$\exists \delta_2 \text{ so } 0 < |x-c| < \delta_2 \Rightarrow |f(x) - L_2| < \varepsilon.$$

So for $0 < |x-c| < \delta_\varepsilon = min\{\delta_1, \delta_2\}$,

$$0 < |L_1 - L_2| = |L_1 - f(x) + f(x) - L_2|$$
$$\leq |f(x) - L_1| + |f(x) - L_2| \text{ by } \triangle$$
$$< \varepsilon + \varepsilon = 2\varepsilon.$$

But $\varepsilon > 0$ was arbitrary.
Hence $|L_1 - L_2| = 0 \Rightarrow L_1 = L_2$. □

Theorem 27 *Let* $f : (a,b) \setminus \{c\} \to \mathbb{R}$*. Then* $\lim_{x\to c} f(x) = L \Leftrightarrow \forall a_n \to c,\ a_n \in (a,b) \setminus \{c\}, f(a_n) \to L$*.*

Note This is applied normally when f is continuous at $x = c$ so $\lim_{x\to c} f(x) = f(c)$. Then $a_n \to c \Rightarrow f(a_n) \to f(c)$.

e.g.
i. $\frac{1}{n} \to 0$ so $e^{\frac{1}{n}} \to e^0 = 1$
ii. $\frac{n+1}{n} \to 1$ so $\sqrt{\frac{n+1}{n}} \to \sqrt{1} = 1$
iii. $\frac{\ln n}{n} \to 0$ so $n^{\frac{1}{n}} = e^{\frac{\ln n}{n}} \to e^0 = 1$.

Proof. ($\Rightarrow$) Let $\lim_{x\to c} f(x) = L$ and $a_n \to c$ ($a_n \neq c$) and $\varepsilon > 0$ be given.
Then $\exists \delta_\varepsilon > 0$ so $0 < |x - c| < \delta_\varepsilon \Rightarrow |f(x) - L| < \varepsilon$.
Replace ε by δ_ε in $a_n \to c \Rightarrow \exists N_\varepsilon \in \mathbb{N}$ so

$$\forall n \geq N_\varepsilon,\ |a_n - c| < \delta_\varepsilon.$$

But we can then replace x by a_n and deduce $|f(a_n) - L| < \varepsilon$.

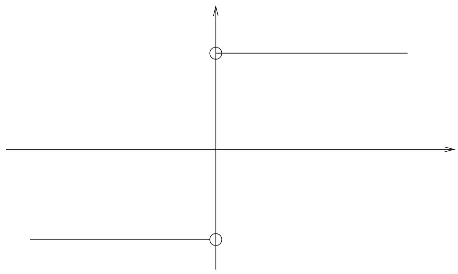
$$\therefore \forall n \geq N_\varepsilon, |f(a_n) - L| < \varepsilon. \text{ Hence } \lim_{n\to\infty} f(a_n) = L.$$

($\Leftarrow$) Assume $\forall a_n \to c, f(a_n) \to L$, but $\lim_{x\to c} f(x) \neq L$(??), i.e. $\exists \varepsilon_0 > 0$ so that $\forall \delta > 0, \exists$ an x_δ so $0 < |x_\delta - c| < \delta$,
but $|f(x_\delta) - L| \geq \varepsilon_0$. Let $\delta = \frac{1}{n}, n = 1, 2, 3, \ldots$.
Then $\exists x_n$ with $0 < |x_n - c| < \frac{1}{n}$ but $|f(x_n) - L| \geq \varepsilon_0$.
But this is enough to have $x_n \to c$ and $f(x_n)$ not approaching L (!!). □

e.g. $f(x) = \frac{|x|}{x}$ $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$. Let $a_n = \frac{(-1)^n}{n}$ then $a_n \to 0$ but

$$f(a_n) = \begin{cases} 1 & \text{n even} \\ -1 & \text{n odd} \end{cases}$$

and $\lim_{n\to\infty} f(a_n) \nexists$ (subsequences tending to different limits). Hence, by Thm 27, $\lim_{x\to 0} f(x) \nexists$.



Theorem 28 Limit Theorem
Let $A = (a, b) \setminus \{c\}$.*Let* $f, g : A \to \mathbb{R}$ *and*

$$\lim_{x \to c} f(x) = L_1 \in \mathbb{R}, \ \lim_{x \to c} g(x) = L_2 \in \mathbb{R}.$$

Then
i. $\lim_{x \to c} f(x) + g(x) = L_1 + L_2 = \lim_{x \to c} f(x) + \lim_{x \to c} g(x)$
ii. $\lim_{x \to c} \alpha f(x) = \alpha L_1$
iii. $\lim_{x \to c} f(x) \cdot g(x) = L_1 \cdot L_2$
iv. If $L_2 \neq 0, \lim_{x \to c} \frac{1}{f(x)} = \frac{1}{L_2}$

e.g. $\lim_{x \to c} x = c$ Let $(\delta_\varepsilon = \varepsilon)$. So, by ii., $\lim_{x \to c} 8x = 8c$.
By iii., $\lim_{x \to c} x^2 = c \cdot c = c^2$ and $\lim_{x \to c} x^3 = \lim_{x \to c} x^2 \cdot x = c^2 \cdot c = c^3$.
And, by induction, $\lim_{x \to c} x^n = c^n, n = 1, 2, 3, ...$ so, by i., $\lim_{x \to c} 3x^2 + x + 1 = 3c^2 + c + 1$.

Proof.
i. Let $\varepsilon > 0$ be given.

$$\exists \delta_1 > 0 \text{ so } 0 < |x - c| < \delta_1 \Rightarrow |f(x) - L_1| < \frac{\varepsilon}{2}.$$

$$\exists \delta_2 > 0 \text{ so } 0 < |x - c| < \delta_2 \Rightarrow |g(x) - L_2| < \frac{\varepsilon}{2}.$$

Let $\delta_\varepsilon = min\{\delta_1, \delta_2\} \Rightarrow$ if $0 < |x - c| < \delta_\varepsilon$ then

$$|f(x) + g(x) - (L_1 + L_2)| = |(f(x) - L_1) + (g(x) - L_2)|$$

$$\leq |f(x) - L_1| + |g(x) - L_2|$$
$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Hence $\lim_{x\to c} f(x) + g(x) = L_1 + L_2$.

ii. If $\alpha = 0$, both sides are zero, so ii. is true. If $c \neq 0$, given $\varepsilon > 0, \exists\delta_\varepsilon > 0$, so

$$0 < |x - c| < \delta_\varepsilon \Rightarrow |f(x) - L_1| < \frac{\varepsilon}{|c|}$$
$$\Rightarrow |cf(x) - cL_1| < \varepsilon.$$
$$\Rightarrow \lim_{x\to c} cf(x) = cL_1$$

iii. Given $\varepsilon > 0, \exists\delta_1 > 0$ so

$$0 < |x - c| < \delta_0 \Rightarrow |f(x) - L_1| < 1$$

$$\Rightarrow |f(x)| = |f(x) - L_1 + L_1| \leq |f(x) - L_1| + |L_1| < 1 + |L_1|.$$
$$\text{and } \exists\delta_1 > 0 \text{ so } |f(x) - L_1| < \varepsilon \text{ for } 0 < |x - c| < \delta_1$$
$$\text{and } \exists\delta_2 > 0 \text{ so } |g(x) - L_2| < \varepsilon \text{ for } 0 < |x - c| < \delta_2.$$

Now let $\delta_\varepsilon = min\{\delta_0, \delta_1, \delta_2\}$ so when $0 < |x - c| < \delta_\varepsilon$:

$$|f(x)g(x) - L_1 \cdot L_2| = |f(x)g(x) - f(x)L_2 + f(x)L_2 - L_1 \cdot L_2|$$

$$\leq |f(x)||g(x) - L_2| + |L_2||f(x) - L_1|$$
$$< (1 + |L_1|)\varepsilon + |L_2|\varepsilon = (1 + |L_1| + |L_2|)\varepsilon.$$

iv. Here $L_1 \neq 0$ so let $\delta_1 > 0$ be such that

$$0 < |x - c| < \delta_1 \Rightarrow |f(x) - L_1| < \frac{|L_1|}{2} \Rightarrow \frac{|L_1|}{2} \leq |f(x)|.$$

Let δ_2 be such that $0 < |x - c| < \delta_2 \Rightarrow |f(x) - L_1| < \varepsilon$.
Then

$$|\frac{1}{f(x)} - \frac{1}{L_1}| = \frac{|f(x) - L_1|}{|f(x)||L_1|}$$

so if $\delta_\varepsilon = min\{\delta_1, \delta_2\}$,

$$< \frac{\varepsilon}{|\frac{|L_1|}{2}| \cdot |L_1|}$$

□

e.g. $\lim_{x\to c} x = c \Rightarrow \lim_{x\to c} \frac{1}{x} = \frac{1}{c}\ \forall c \neq 0.$

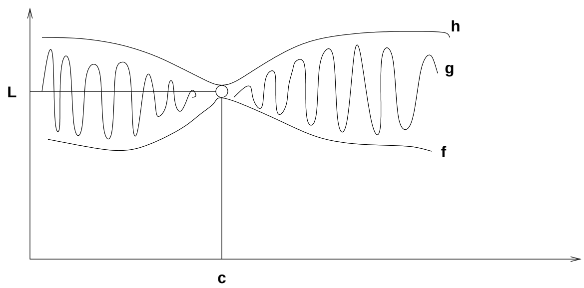
e.g. If P is a polynomial, $\lim_{x\to c} P(x) = P(c)\ \forall c \in \mathbb{R}$.

e.g. If R is a rational function, $R(x) = \frac{P(x)}{Q(x)}, P, Q$ polynomials, then

$$Q(c) \neq 0 \Rightarrow \lim_{x\to c} R(x) = \frac{\lim_{x\to c} P(x)}{\lim_{x\to c} Q(x)} = \frac{P(c)}{Q(c)} = R(C).$$

Thus $R(x)$ is **continuous** except when $Q(c) = 0$.

e.g. $\lim_{x\to 1} \frac{x^2+x-1}{x+3} = \frac{1^2+1-1}{1+3} = \frac{1}{4}$ since $1+3 \neq 0$.



Theorem 29 Sandwich

Let $\forall x \in A,\ f(x) \leq g(x) \leq h(x),\ \lim_{x\to c} f(x) = \lim_{x\to c} h(x) = L$.
Then $\lim_{x\to c} g(x) = L$ *also.*

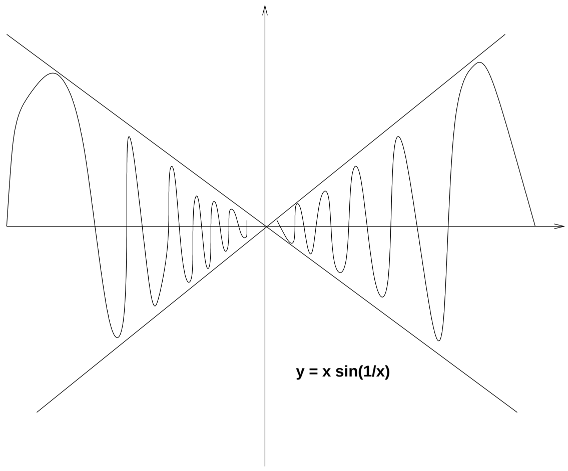
Proof. Let $\varepsilon > 0$ be given.

$$\exists \delta_1 > 0 \text{ so } 0 < |x-c| < \delta_1 \Rightarrow L - \varepsilon < f(x) < L + \varepsilon$$

and

$$\exists \delta_2 > 0 \text{ so } 0 < |x-c| < \delta_2 \Rightarrow L - \varepsilon < h(x) < L + \varepsilon.$$

So let $\delta_\varepsilon = min\{\delta_1, \delta_2\}$.
Then $0 < |x - c| < \delta_\varepsilon \Rightarrow L - \varepsilon < f(x) \leq g(x) \leq h(x) < L + \varepsilon$.
$\Rightarrow L - \varepsilon < g(x) < L + \varepsilon$
$\Rightarrow |g(x) - L| < \varepsilon$.
Hence $\lim_{x \to c} g(x) = L$ also. □



e.g. $g(x) = x \sin(\frac{1}{x})$ at c = 0.

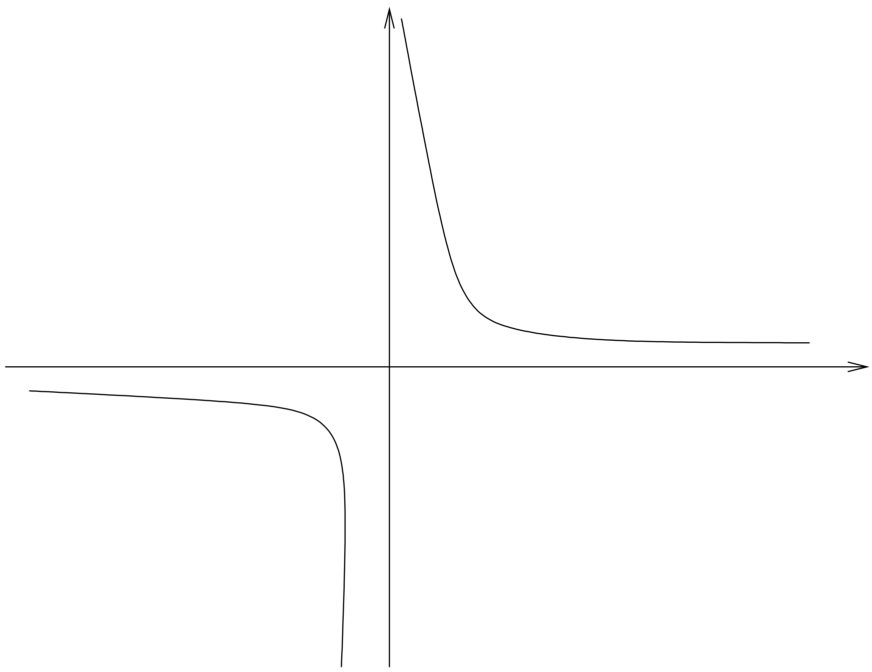
Then $-1 \leq \sin(\frac{1}{x}) \leq 1 \ \forall x \neq 0$. if $x \geq 0$ then multiply by x to get $-|x| = -x \leq x \sin 1/x \leq x = |x|$. Similarly for $x < 0$.

$$\Rightarrow -|x| \leq x \sin(\frac{1}{x}) \leq |x|.$$

So, since $\lim_{x \to c} |x| = 0 = \lim_{x \to c}(-|x|)$, we get

$$\lim_{x \to 0} x \sin(\frac{1}{x}) = 0.$$

Other limits of functions



e.g. $y = \frac{1}{x}$.

1. $\lim\limits_{x\to\infty} \dfrac{1}{x} = 0.$

2. $\lim\limits_{x\to-\infty} \dfrac{1}{x} = 0.$

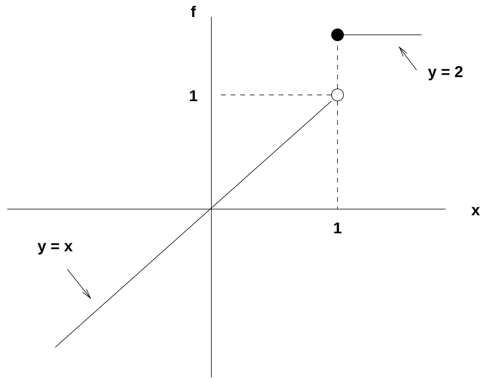
3. $\lim\limits_{x\to 0^+} \dfrac{1}{x} = +\infty$

4. $\lim\limits_{x\to 0^-} \dfrac{1}{x} = -\infty$

5.

$$f(x) = \begin{cases} x, & x < 1, \\ 2, & x \geq 1 \end{cases}$$

$$\lim_{x\to 1^+} f(x) = 2 = f(1)$$



6. $\lim_{x\to 1^-} f(x) = 1 \neq f(1)$ and $\lim_{x\to 1} f(x) \nexists$.
7. If we had these limits we could express:

$$\lim_{x\to 0^+} \frac{1}{x} = +\infty$$

8 $\lim_{x\to 0^-} \frac{1}{x} = \infty$

These all fit into our "we can make $f(x)$ arbitrarily close to something (or arbitrarily large and positive or negative) provided we choose x sufficiently close (or sufficiently large or large and negative or close and on one side) to something.

e.g. 5 $\lim_{x\to c^+} f(x) = L$ if $\forall \varepsilon > 0, \exists \delta_\varepsilon > 0$ such that

$$c < x < c + \delta_\varepsilon \;\Leftrightarrow\; 0 < x - c < \delta_\varepsilon \Rightarrow |f(x) - L| < \varepsilon.$$

e.g.6 $\lim_{x\to c^-} f(x) = L$ if $\forall \varepsilon > 0, \exists \delta_\varepsilon > 0$ such that

$$c - \delta_\varepsilon < x < c \;\Leftrightarrow\; 0 < c - x < \delta_\varepsilon \;\Rightarrow |f(x) - L| < \varepsilon.$$

Some of these limits are connected.

Theorem 30 $\lim_{x\to c} f(x) = L \Leftrightarrow \lim_{x\to c+} f(x) = L = \lim_{x\to c^-} f(x)$.

Proof. $(\Rightarrow)$ $\forall \varepsilon > 0, \exists \delta_\varepsilon > 0$ so $0 < |x-c| < \delta_\varepsilon \Rightarrow |f(x) - L| < \varepsilon$.

But

$$c < x < c + \delta_\varepsilon \Rightarrow 0 < x - c < \delta_\varepsilon \Rightarrow 0 < |x-c| < \delta_\varepsilon.$$

Hence $\lim\limits_{x \to c^+} f(x) = L$.

$(\Leftarrow)$

1. Given $\varepsilon > 0, \exists \delta_1 > 0$ so $0 < x - c < \delta_1 \Rightarrow |f(x) - L| < \varepsilon$.

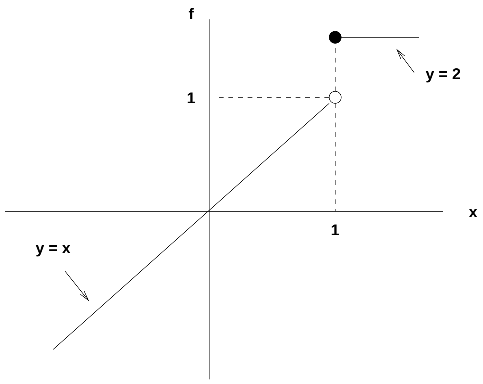
$*$

2. Given $\varepsilon > 0, \exists \delta_2 > 0$ so $0 < c - x < \delta_2 \Rightarrow |f(x) - L| < \varepsilon$.

So let $\delta_\varepsilon = min\{\delta_1, \delta_2\}$ and x satisfy $0 < |x-c| < \delta_\varepsilon$.
If $x > c$, use 1. to get $0 < |x-c| = x - c < \delta_\varepsilon \leq \delta_1 \Rightarrow *$
If $x < c$, use 2. to get $0 < |c-x| = c - x < \delta_\varepsilon \leq \delta_2 \Rightarrow *$
Hence $\lim\limits_{x \to c} f(x) = L$. □

This is very useful: If $f(x) = g(x)$ on (c, b), then $\lim\limits_{x \to c^+} f(x) = \lim\limits_{x \to c^+} g(x)$ (and vice versa for $x \to c^-$)

e.g.



$$f(x) = \begin{cases} x, \ x < 1, \\ 2, \ x \geq 1 \end{cases}$$

Let $g(x) = x \;\; \forall x \in \mathbb{R}$ and $h(x) = 2 \;\; \forall x \in \mathbb{R}$.

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} h(x) = \lim_{x\to 1^+} 2 = 2 = f(1),$$

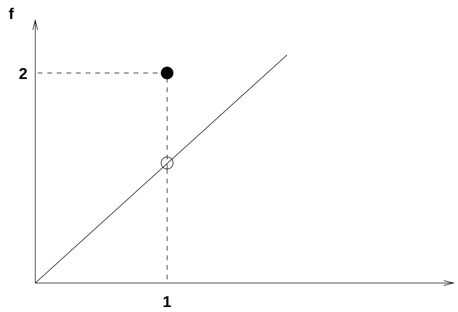
and

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} g(x) = g(1) = 1.$$

So we say f is **continuous on the right** (but not on the left).

e.g. $h(x) = x \;\; \forall x \in \mathbb{R}$,

$$f(x) = \begin{cases} x, & x \neq 1 \\ 2, & x = 1 \end{cases}$$



Then $\lim_{x\to 1} f(x) = \lim_{x\to 1} h(x) = h(1) = 1 \neq f(1)$.

e.g. $y = f(x) = \frac{x+4}{x-5}$, $\lim_{x\to 5-} f(x) = -\infty$.
Meaning $f(x)$ gets arbitrarily large and negative provided x is less than 5 but sufficiently close to 5.
Changing variables might help:

$$h = 5 - x \text{ so } x < 5 \Leftrightarrow h > 0, \;\; x \to 5- \Leftrightarrow h \to 0+,$$

$$f(x) = \frac{5 - h + 4}{-h} = \frac{9 - h}{-h} = 1 - \frac{9}{h} = g(h).$$

We need a definition.

$$\lim_{x\to c-} f(x) = -\infty \Leftrightarrow \; \forall M > 0, \exists \delta_M > 0 \text{ so } c - \delta_M < x < c \Rightarrow f(x) < -M.$$

OR

$$\lim_{h\to 0+} g(h) = -\infty \Leftrightarrow \; \forall M > 0, \exists \delta_M > 0 \text{ so } 0 < h < \delta_M \Rightarrow g(h) < -M.$$

So, given $M > 0$, we want $g(h) < -M$

$$\Leftrightarrow 1 - \frac{9}{h} < -M$$

$$\Leftrightarrow 1 + M < \frac{9}{h}$$

$$\Leftrightarrow \frac{1}{M+1} > \frac{h}{9}$$

$$\Leftrightarrow \frac{9}{M+1} > h$$

So let $\delta_M = \frac{9}{M+1}$.

So now vertical asymptotes have been tamed. How about the horizontal asymptotes?

Again: $f(x) = \frac{x+4}{x-5} = \frac{1+\frac{4}{x}}{1-\frac{5}{x}} \to 1$ as $x \to \pm\infty$
Meaning we can make $f(x)$ arbitrarily close to 1 provided we choose x sufficiently large and positive or sufficiently large and negative.
OR

$$\lim_{x\to\infty} f(x) = L \Leftrightarrow \ \forall\varepsilon > 0, \exists M > 0 \text{ so } \forall x \geq M, |f(x) - L| < \varepsilon$$

and

$$\lim_{x\to-\infty} f(x) = L \Leftrightarrow \ \forall\varepsilon > 0, \exists M > 0 \text{ so } \forall x \leq -M, |f(x) - L| < \varepsilon.$$

e.g. $\lim\limits_{x\to\infty} \dfrac{x+4}{x-5} = 1$. Given $\varepsilon > 0$, want $|\frac{x+4}{x-5} - 1| < \varepsilon$

$$\Leftrightarrow |\frac{x+4-x+5}{x-5}| < \varepsilon$$

$$\Leftrightarrow \frac{9}{|x-5|} < \varepsilon$$

Let $M_1 = 6$ so $x \geq M_1 \Rightarrow |x-5| = x-5$.
So we need

$$\frac{9}{x-5} < \varepsilon \Leftrightarrow \frac{x-5}{9} > \frac{1}{\varepsilon} \Leftrightarrow x > 5 + \frac{9}{\varepsilon}.$$

So let $M = 6 + \frac{9}{\varepsilon}$.
Then

$$x \geq M \Rightarrow x \geq 6,\ x \geq 6 + \frac{9}{\varepsilon} > 5 + \frac{9}{\varepsilon} \Rightarrow |f(x) - 1| < \varepsilon.$$

Hence $\lim_{x\to\infty} \frac{x+4}{x-5} = 1$.

9 Indeterminate Forms and L'Hopital's Rule

Theorem 31 *If $f(a) = g(a) = 0$ and $f'(a), g'(a)$ exist with $g'(a) \neq 0$, then*

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)}.$$

e.g. $f(x) = \sin x, f(0) = \sin 0 = 0, f'(x) = \cos x,$
$g(x) = x, g(0) = 0, g'(x) = 1$

$$\Rightarrow \lim_{x\to 0} \frac{\sin x}{x} = \frac{f'(0)}{g'(0)} = \frac{\cos 0}{1} = \frac{1}{1} = 1.$$

Proof.

$$\frac{f'(a)}{g'(a)} = \frac{\lim_{h\to 0} \frac{f(a+h) - f(a)}{h}}{\lim_{h\to 0} \frac{g(a+h) - g(a)}{h}}$$
$$= \lim_{h\to 0} \frac{f(a+h) - f(a)}{g(a+h) - g(a)}$$
$$= \lim_{h\to 0} \frac{f(a+h)}{g(a+h)}$$
$$x - a = h$$

$$= \lim_{x\to a} \frac{f(x)}{g(x)}.$$ □

More general form:

Theorem 32 *If $f(a) = g(a) = 0$ and $g'(x) \neq 0$ for $x \neq a$. Then if the RH limit exists:*

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}.$$

e.g. $\frac{\cos x - 1}{x^2}$ is $\frac{0}{0}$ at $x = 0$, but

$$\frac{(\cos x - 1)''}{(x^2)'} = \frac{(-\sin x)'}{(2x)'} = \frac{-\cos x}{2} \to -\frac{1}{2}.$$

Theorem 33 *If $f(x), g(x) \to \infty$ as $x \to a\pm$ and $\lim_{x\to a\pm} \frac{f'(x)}{g'(x)} = L\exists$, then $\lim_{x\to a\pm} \frac{f(x)}{g(x)} = L$ also, and a could be $\pm\infty$.*

e.g. $\lim_{x\to 5-} \frac{x+4}{x-5}$ is not suitable for l'Hopital's Rule.

e.g. $\lim_{x\to\infty} \frac{x+4}{x-5} = \lim_{x\to\infty} \frac{(x+4)'}{(x-5)'} = \lim_{x\to\infty} \frac{1}{1} = 1$.

e.g. $\lim_{x\to\infty} x\sin(\frac{1}{x})$.
Let $x = \frac{1}{h}$ so $x \to \infty \Leftrightarrow h \to 0+$ $\lim_{h\to 0+} \frac{\sin h}{h} = \lim_{h\to 0+} \frac{\cos h}{1} = \cos 0 = 1$.

e.g. $\lim_{x\to\infty} (1 + \frac{1}{x})^x$.
Let $f(x) = (1 + \frac{1}{x})^x \Rightarrow \ln f(x) = x\ln(1 + \frac{1}{x})$.

$$\Leftrightarrow \lim_{x\to\infty} \ln f(x) = \lim_{h\to 0+} \frac{\ln(1+h)}{h} = \lim_{h\to 0+} \frac{1}{(1+h)1} = 1.$$

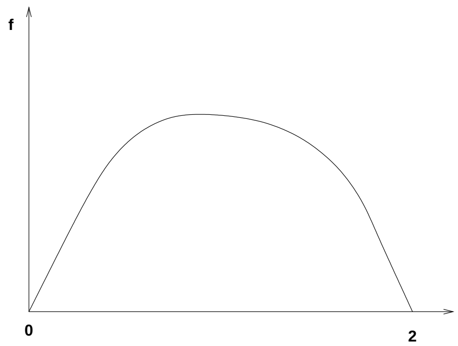
Hence $\lim_{x\to\infty} f(x) = \exp(\lim_{x\to\infty} \ln f(x)) = e^1 = e$.
Hence $\lim_{x\to\infty} (1 + \frac{1}{x})^x = e$.

10 Continuous Functions

Definition
f is **continuous on the right** at a if $\lim_{x \to a+} f(x) = f(a)$.
f is **continuous on the left** at b if $\lim_{x \to b-} f(x) = f(b)$.
$f : a, b \to \mathbb{R}$ is **continuous** (on $[a, b]$)if it is continuous on (a, b) and on the right at a and on the left at b.

e.g. $f(x) = x(2 - x)$

$$x \in [0, 2], f : [0, 2] \to \mathbb{R}.$$



f is continuous on $[0, 2]$.

Theorem 34 *If $a < b, f, g : [a, b] \to \mathbb{R}$ continuous and $\alpha, \beta \in \mathbb{R}$ then $\alpha f + \beta g$ is continuous on $[a, b]$.*

Proof. If $a < c < b, \lim_{x \to c} \alpha f(x) + \beta g(x) = \alpha \lim_{x \to c} f(x) + \beta \lim_{x \to c} g(x)$

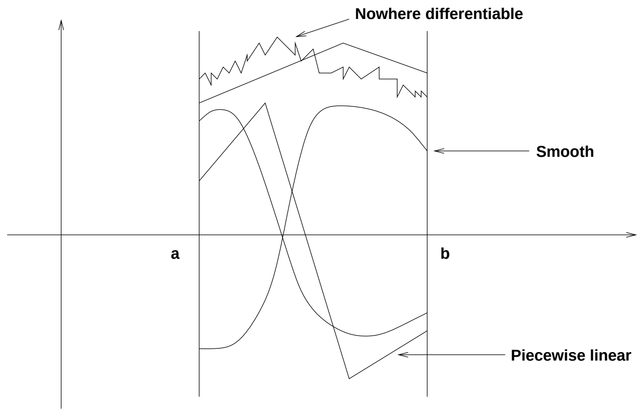
$$= \alpha f(c) + \beta g(c)$$

$$= (\alpha f + \beta g)(c).$$

$\Rightarrow \alpha f + \beta g$ is continuous at c.
Similarly, it is continuous on the right at a and on the left at b. □

Note The set $C[a, b] = \{f : [a, b] \to \mathbb{R} : \text{f is continuous}\}$ is, by Thm **??** a vector space over $\mathbb{R}$. It has ∞ dimension.

Note $C[a, b]$ has many "nice" functions and many "bad" ones.



Theorem 35 Continuous functions on closed intervals are bounded
Let $f \in C[a,b]$. Then f is **bounded**, *i.e.* $\exists M$ *so* $|f(x)| \leq M \forall x \in [a,b]$.

Proof. Because $\lim_{x \to a+} f(x) = f(a)$ $\exists \delta_1 > 0$ so $a \leq x < a + \delta_1 \Rightarrow |f(x) - f(a)| < 1$.
$\Rightarrow f(x) < f(a) + 1$ so f is bounded on $[a, a + \frac{\delta_1}{2}]$.
Let $S = \{\eta : a < \eta \leq b$ and f is bounded on $[a, \eta]\}$.
1. $a + \frac{\delta_1}{2} \in S \Rightarrow S \neq 0$.
2. b is an upper bound for S.
3. If $\eta \in S$ and $a < x \leq \eta$ then $x \in S$.
Let $\alpha = lub(S)$. Then $a < \alpha \leq b$.
If $\alpha < b$, then, since f is continuous at $\alpha, \exists \delta_2 > 0$ so

$$\alpha - \delta_2 < x < \alpha + \delta_2 \Rightarrow |f(x) - f(\alpha)| < 1 \Rightarrow f(x) < 1 + f(\alpha).$$

Thus f is bounded on $[\alpha - \frac{\delta_2}{2}, \alpha + \frac{\delta_2}{2}]$. But it is bounded also on $[a, \alpha - \frac{\delta_2}{2}]$ by definition of α.
Therefore, f is bounded on $[a, \alpha + \frac{\delta_2}{2}]$ so $\alpha + \frac{\delta_2}{2} (> \alpha) \in S$, contradicting the definition of α (!!).
Hence $\alpha = b$.
Since $\lim_{x \to b-} f(x) = f(b), \exists \delta_3 > 0$ so $b - \delta_3 < x \leq b \Rightarrow f(x) < 1 + f(b)$.
So f is bounded on $[a, b - \frac{\delta_3}{2}]$ and $[b - \frac{\delta_3}{2}, b] \Rightarrow f$ is bounded on $[a,b]$. □

e.g. $f(x) = \frac{1}{1-x}$ on $[0,1)$ has $\alpha = lub(x) = 1$, but f is not bounded on $[0,1)$.

Theorem 36 Continuous functions on closed intervals attain their bounds
Let $f\colon [a,b] \to \mathbb{R}$ be continuous and let $M = lub\{f(x) : x \in [a,b]\}$. Then $\exists \eta \in [a,b]$ such that $f(\eta) = M$.

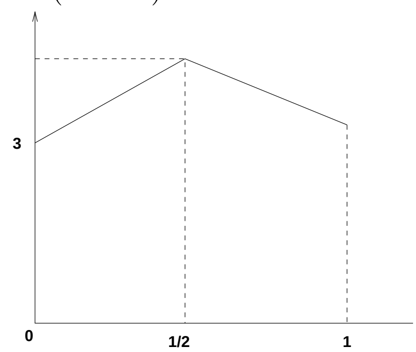
Proof. Suppose $f(x) < M \ \forall x \in [a,b]$. Let $k > 0$ be a given (large) number and choose $\varepsilon > 0$ so $0 < \varepsilon < \frac{1}{k}$. Then, for some $x \in [a,b]$,

$$M - \varepsilon < f(x) < M \Rightarrow M - f(x) < \varepsilon \Rightarrow \frac{1}{M - f(x)} > \frac{1}{\varepsilon} > k.$$

Thus, $g(x) = \frac{1}{M-f(x)}$ is continuous and unbounded on $[a,b]$ (!!).
Hence, there must be an $\eta \in [a,b]$ with $f(\eta) = M$. □

e.g. Let $m = glb\{f(x) : x \in [a,b]\}$.
Then $\exists \beta \in [a,b]$ so $f(\beta) = m$. To see this, consider $g(x) = -f(x)$.

e.g. $f(x) = 4 - |2x - 1|$ on $[0, 1]$:
If $2x - 1 > 0 (\Leftrightarrow x > \frac{1}{2}), f(x) = 4 - (2x - 1) = 5 - 2x$.
If $2x - 1 \leq 0, f(x) = 4 + (2x - 1) = 3 + 2x$.



Then $\eta = \frac{1}{2}$ and $\beta = 0$ or 1 and $f'(\eta) \nexists$.

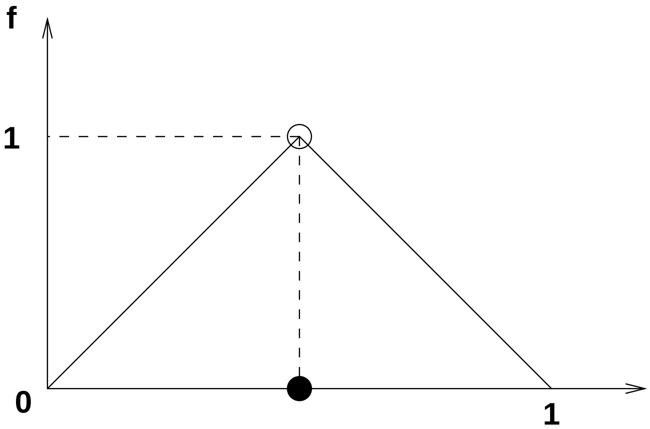
Theorem 37 Intermediate Value Theorem
Let $f\colon [a,b] \to \mathbb{R}$ be continuous and ξ satisfy $f(a) \leq \xi \leq f(b)$. Then for some $\eta \in [a,b], f(\eta) = \xi$.

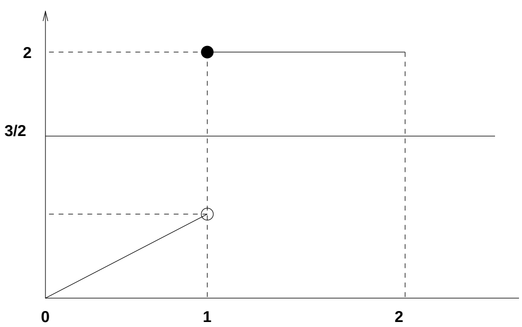
Proof. Let $S = \{x\colon a \leq x \leq b \text{ and } f(x) \leq \xi \text{ and } \forall y \in [a,x), f(y) \leq \xi\}$.
Then $S \subset [a,b]$ is bounded above by b.
Let $\eta = lub(S)$. Then $a \leq \eta \leq b$.
Claim: $f(\eta) = \xi$.

1. If $f(a) = \xi$ on $f(b) = \xi$ we are done, so assume $f(a) < \xi < f(b)$.
2. There is a sequence (x_n) in S with $x_n \to \eta$. Because $f(x_n) \leq \xi \Rightarrow \lim\limits_{n\to\infty} f(x_n) = f(\eta) \leq \xi$, since f is continuous.
3. Since $\xi < f(b), \eta = b$ is false. Thus $\eta < b$. If $f(\eta) < \xi$, we can find an interval $[\eta, \eta + \delta)$ so $f(x) \leq \xi$ on this interval as follows: since f is continuous on the right at η, if $\varepsilon = \xi - f(\eta), \exists\delta > 0$ so $\eta \leq x < \eta + \delta \Rightarrow f(\eta) - \varepsilon < f(x) < f(\eta) + \varepsilon = f(\xi)$.
$\therefore$ $f(x) \leq \xi$ on $[a, \eta]$ and $[\eta, \delta + \eta) \Rightarrow \eta + \frac{\delta}{2} \in S$, a contradiction since $\eta = lub(S)$. Hence $f(\eta) = \xi$. □

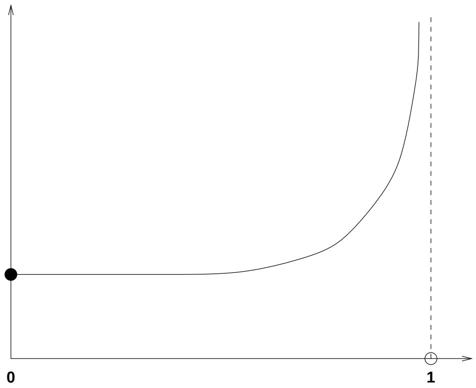
So, when $f\colon [a, b] \to \mathbb{R}$ is continuous, f is bounded and attains its bounds, and has the intermediate value property.

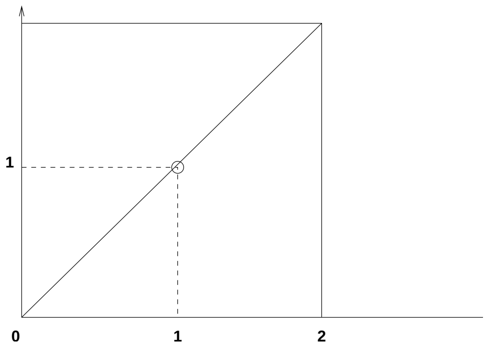
Note (i) The same applies to **any closed subinterval** $[\alpha, \beta] \subset [a, b]$.
Note (ii) The hypothesis of **continuity is essential**.





Note(iii) The domain must be a closed, bounded, interval.





Definition We say $f\colon (a,b) \to \mathbb{R}$ is **differentiable at** $a \in (c,b)$ if

$$\lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$$

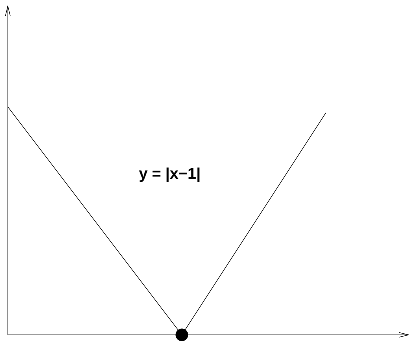
exists in $\mathbb{R}$. If so, we write its value $f'(a)$ and call this the **derivative** of f at $x = a$.

e.g. $f(x) = |x-1|$ is not differentiable at $x = 1$:

$$\lim_{h\to 0+} \frac{f(1+h)-f(1)}{h} = \lim_{h\to 0+} \frac{h-0}{h} = 1$$

$$\lim_{h\to 0-} \frac{f(1+h)-f(1)}{h} = \lim_{h\to 0-} \frac{-h}{h} = -1$$

$$\therefore f'(1) \nexists.$$



Theorem 38 *If f is differentiable at $x = a$, then f is continuous at $x = a$.*

Proof.

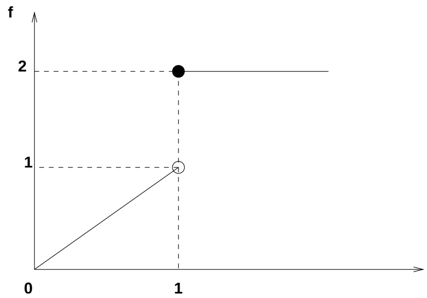
$$\lim_{x \to a} f(x) = \lim_{x \to a} [\frac{f(x) - f(a)}{x - a}(x - a) + f(a)]$$

$$x - a = h$$

$$= \lim_{h \to 0} [\frac{f(a + h) - f(a)}{h} h + f(a)]$$
$$= f'(a) \cdot 0 + f(a) = f(a).$$

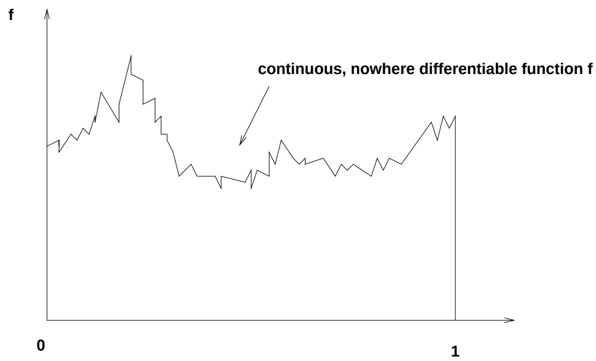
□

e.g. $f(1) \nexists \therefore f$ is not continuous at $x = 1$.



e.g. $f(x) = |x - 1|$ is continuous but not differentiable at $x = 1$.

e.g. Most continuous functions are not differentiable anywhere!



Theorem 39 *Let* $f\colon (a,b) \to \mathbb{R}$ *be differentiable at* $\xi \in (a,b)$.
i. If $f'(\xi) > 0, \exists \delta > 0$ *so* $f(x) < f(\xi) < f(y)\ \forall x, y$ *with*

$$\xi - \delta < x < \xi < y < \xi < \delta.$$

ii. If $f'(\xi) < 0, \exists \delta > 0$ *so* $f(x) > f(\xi) > f(y)\ \forall x, y$ *with*

$$\xi - \delta < x < \xi < y < \xi + \delta.$$

iii. If f *has a local max or min at* ξ *then* $f'(\xi) = 0$.

Proof. (i). Let $\varepsilon = \frac{f'(\xi)}{2}$.
Since $f'(\xi) = \lim_{x \to \xi} \dfrac{f(x) - f(\xi)}{x - \xi}, \exists \delta > 0$ such that

$$0 < |x - \xi| < \delta,\ \left|\frac{f(x) - f(\xi)}{x - \xi} - f'(\xi)\right| < \frac{f'(\xi)}{2}.$$

If y satisfies $\xi < y < \xi + \delta \Rightarrow 0 < y - \xi < \delta$ and thus

$$\Rightarrow -\frac{f'(\xi)}{2} < \frac{f(y) - f(\xi)}{y - \xi} - f'(\xi) < f'(\xi)$$

$$\Rightarrow 0 < \frac{f'(\xi)}{2} < \frac{f(y) - f(\xi)}{y - \xi} \Rightarrow f(\xi) < f(y).$$

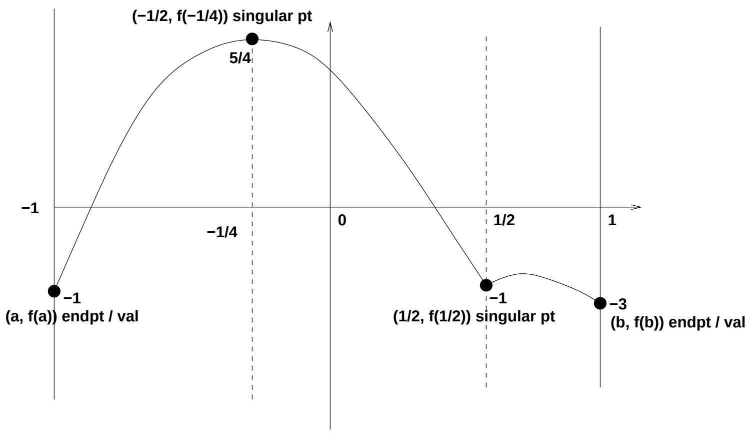
If x satisfies $\xi - \delta < x < \xi$ then

$$0 < \xi - x < \delta \Rightarrow 0 < \frac{f'(\xi)}{2} < \frac{f(\xi) - f(x)}{\xi - x} = \frac{f(x) - f(\xi)}{x - \xi} \Rightarrow f(x) < f(\xi).$$

(ii). $\varepsilon = -\frac{f'(\xi)}{2}$.
(iii). If ξ is a local max or min then $f'(\xi) = 0$, since otherwise we would get a contradiction, by (i) or (ii). □

Corollary If $f\colon [a, b] \to \mathbb{R}$ is continuous, its global max and min occur among
a. endpoints a, b
b. points where $f'(x)$ does not exist (singular points).
c. points where $f'(x) = 0$ (critical points).

e.g. $f(x) = |2x - 1| - 4x^2$ on [-1, 1].



1. If $2x - 1 \geq 0$ ($\Leftrightarrow x \geq \frac{1}{2} \Leftrightarrow \frac{1}{2} \leq x \leq 1$),

$$f(x) = 2x - 1 - 4x^2$$

$$f(\frac{1}{2}) = -1, \quad f(1) = -3.$$

$$f'(x) = 2 - 8x = 0 \Rightarrow x = \frac{1}{4}$$

So no critical points in this section $[\frac{1}{2}, 1]$.

2. If $2x-1 \le 0$ ($\Leftrightarrow -1 \le x \le \frac{1}{2}$),

$$f(x) = -(2x-1) - 4x^2 = -2x + 1 - 4x^2$$

$$f(-1) = -1, \quad f(\frac{1}{2}) = -1$$

$$f'(x) = -2 - 8x = 0 \Rightarrow x = -\frac{1}{4} \text{ and } -1 \le -\frac{1}{4} \le \frac{1}{2}$$

So $(-\frac{1}{4}, f(-\frac{1}{4}))$ is a critical point.

3.

$$f'_+(\frac{1}{2}) = \lim_{h\to 0} \frac{f(\frac{1}{2}+h) - f(\frac{1}{2})}{h} = (2x-1-4x^2)'|_{x=\frac{1}{2}} = (2-8x)|_{x=\frac{1}{2}} = -2.$$

$$f'_-(\frac{1}{2}) = (-2x+1-4x^2)'|_{x=\frac{1}{2}} = (-2-8x)|_{x=\frac{1}{2}} = -6 \ne f'_+(\frac{1}{2}).$$

Hence $(\frac{1}{2}, f(\frac{1}{2}))$ is a singular point/value, the only one.

4. The global max is the largest of $\{f(-1), f(-\frac{1}{4}), f(\frac{1}{2}), f(1)\}$, i.e. $\frac{5}{4}$.

Definition We say f is **differentiable on the right at a** in $\mathbb{R}$ if $\lim_{h\to 0+} \frac{f(a+h)-f(a)}{h}$ $\exists$ as a real number.
We say f is **differentiable at a** if $\lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$ $\exists$.

Theorem 40 *If $f: \mathbb{R} \to \mathbb{R}$ is differentiable on the right at a then f is continuous on the right at a.*

Proof. See your first year Calculus notes or prove this as an exercise □

e.g. $f(x) = |x|$ is not differentiable at $x = 0$.

It is instructive to consider Newtonian quotients other than the one given in the above definition, e.g.

$$\lim_{h\to 0} \frac{f(a+h) - f(a-h)}{2h} = f'(a)$$

if f is differentiable at $x = a$. However, if $f(x) = |x|$ and $a = 0$, the limit on the left exists, i.e. we may assign a slope sensibly to a function with no derivative at a given point.

Exercise Examine the expressions

$$\frac{f(a+\lambda h) - f(a-(1-\lambda)h)}{h} \quad \text{for } 0 \le \lambda \le 1.$$

Then $\lambda = 1$ is the usual quotient, $\lambda = \frac{1}{2}$ is equivalent to the one given above, and $\lambda = 0$ is equivalent to the usual derivative when the full limit $h \to 0$ is taken.

Theorem 41 *Let f, g be functions differentiable at $a \in \mathbb{R}$ with derivatives $f'(a), g'(a)$ respectively, and let $C \in \mathbb{R}$. Then*
(1) $f + g$ is differentiable at a and $(f+g)'(a) = f'(a) + g'(a)$
(2) $f \cdot g$ is differentiable at a and $(f \cdot g)'(a) = f(a)g'(a) + g'(a) \cdot g(a)$
(3) $c \cdot f$ is differentiable at a and $(c \cdot f)'(a) = cf'(a)$
(4) If $g(a) \neq 0, \frac{1}{g}$ is differentiable at a and $\frac{1}{g}'(a) = -\frac{g'(a)}{g^2(a)}$.

Proof. See your first year Calculus notes. ☐

Taking derivatives, because of the previous theorem and the chain rule, the next theorem, appears to be almost a mechanical procedure and becomes tiresome, especially for higher derivatives of complex expressions. There are several excellent computer languages which know these theorems and take the pain out of differentiation.

Theorem 42 Chain Rule
Let $f\colon \mathbb{R} \to \mathbb{R}$, $g\colon \mathbb{R} \to \mathbb{R}$ be functions. Let $\varphi = g \circ f$ be the composite function. Let f be differentiable at a point $a \in \mathbb{R}$ and let g be differentiable at $f(a) = b$. Then φ is differentiable at a and $\varphi'(a) = g'(f(a)) \cdot f'(a)$.

Proof. Define a function $\theta\colon \mathbb{R} \to \mathbb{R}$ by

$$\theta(y) = \begin{cases} g'(b) & \text{if } y = b \\ \frac{g(y)-g(b)}{y-b} & \text{if } y \neq b \end{cases}$$

Then $\displaystyle\lim_{y \to b} \theta(y) = \lim_{y \to b} \frac{g(y)-g(b)}{y-b} = g'(b) = \theta(b)$.

Therefore θ is continuous at $b \in \mathbb{R}$. □

Claim $\varphi(x) - \varphi(a) = \theta(f(x))(f(x) - b)\ \forall x \in \mathbb{R}$. (1)

Proof. If $f(x) = b$ then

$$\text{LHS } = g(f(x)) - g(f(a)) = g(b) - g(b) = 0 = \text{ RHS.}$$

If $f(x) \neq b$ then

$$\text{RHS } = \frac{g(f(x)) - g(b)}{f(x) - b}(f(x) - b) = \text{ LHS.}$$

From (1),

$$\lim_{x \to a} \frac{\varphi(x) - \varphi(a)}{x - a} = \lim_{x \to a} \theta(f(x)) \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \theta(f(a)) \cdot f'(a)$$

where the first limit on the right hand side follows because of the continuity of f and θ and the second because of the differentiability of f at a.
Hence the limit given by the left hand side exists and so

$$\varphi'(a) = g'(b) \cdot f'(a) \text{ since } \theta(f(a)) = \theta(b).$$

□

Exercise Find the global maxima and minima for
i. $2x^2 - 3|x|$ on [-1, 1]
ii. $|2x - 1| + |3x - 2| + x^2$ on [-1, 1]
iii. $(x - 1)|x - 2|$ on [0, 3]

Theorem 43 Rolle's Theorem
Let $f\colon [a, b] \to \mathbb{R}$ be continuous and let f be differentiable at $\xi\ \forall \xi \in (a, b)$. If $f(a) = f(b)$ then there is at least one point η in (a, b) with $f'(\eta) = 0$.

Proof. (Compare your first year calculus notes) Since f is continuous, it has and attains its glb, m, say, and lub, M, say, on $[a, b]$. Then either
i. $m < f(a) = f(b)$
ii. $f(a) = f(b) < M$
iii. $M = m = f(a) = f(b)$

There are no other possibilities.
In case iii., $f(x) = M\ \forall x \in (a, b)$ and so $f'(x) = 0\ \forall$ such x.
In case i., $\exists \eta \in (a, b)$ such that $f(\eta) = m$. Thus, $\eta \neq a$ or b and so $f'(\eta)\ \exists$.

$$\therefore f'(\eta) = 0.$$

The proof in case ii. is similar to case i. □

Theorem 44 Mean Value Theorem
Left $f\colon [a, b] \to \mathbb{R}$ be continuous and let $f'(\xi)$ exist at each point $\xi \in (a, b)$. Then $\exists \eta \in (a, b)$ such that $f(b) - f(a) = (b - a)f'(\eta)$.

Proof. Apply Rolle's Theorem to the function

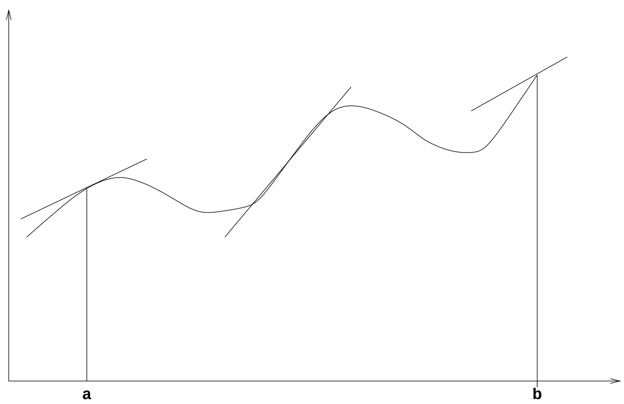
$$\phi(x) = f(x) - f(a) - \frac{f(b) - f(a)}{b - a}(x - a) \text{ on } [a, b].$$

□

You will be very familiar with the two previous theorems but they are stated here because of their fundamental importance in the development of Taylor's Theorem.

10.1 Repeated Derivatives

Let $f\colon [a, b] \to \mathbb{R}$ be a function. We say f is differentiable on $[a, b]$ if $f'(\xi)\exists\ \forall \xi \in (a, b)$ and the left hand derivative exists (as a real number) at b and the right hand limit derivative exists at a.



In this way we form a new function $x \to f'(x)$ on $[a, b]$. It may or may not be continuous, differentiable, and so on. If its derivative exists at a point $\xi \in (a, b)$ we call it $f''(\xi)$ or $f^{(2)}(\xi)$ where

$$f^{(0)}(\xi) = f(\xi)$$

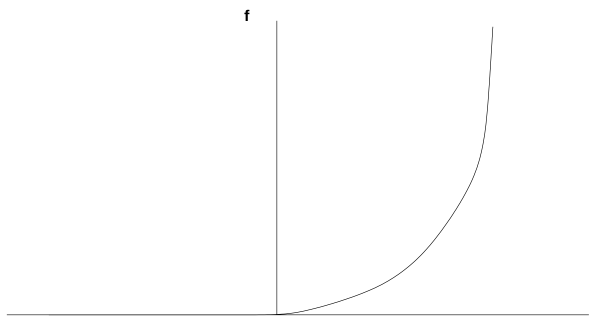
$$f^{(1)}(\xi) = f'(\xi)$$

$$f^{(2)}(\xi) = f''(\xi)$$

Exercise $f''(\xi) = \lim_{h \to 0} \frac{f(\xi + h) - 2f(\xi) + f(\xi - h)}{h^2}$

In general we may define $f^{(n+1)}(x) = (f^{(n)}(x))'$, the $(n+1)$th derivative whenever the derivative of the nth derivative exists. The number of continuous derivatives that exist for a given function is often taken as a measure of its "smoothness". Indeed functions with an infinite number of derivatives (such as $x^2 + 3x + 7$ or $\sin x$) are said to be "smooth". Care is needed because functions which look smooth might be only superficially so (examples below). Traditionally scientists have demanded "smooth" functions. However, functions with continuous second derivatives only are currently accepted as a better family for performing many operations, **e.g.** interpolation, surface fitting, solving differential equations numerically.

e.g.



$$f(x) = \begin{cases} 0 & \text{if } x \le 0 \\ x^2 & \text{if } x > 0 \end{cases}$$

Then

$$f'(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ 2x & \text{if } x \geq 0 \end{cases}$$

and $f'(x)$ is continuous on $\mathbb{R}$. However, $f''(0)$ does not exist. If the second line of the definition of f used x^3 instead of x^2 then $f''(x)$ would exist and be continuous on $\mathbb{R}$ but $f'''(0) = f^{(3)}(0)$ would not exist.

Problem Let $a < b$, and let $\alpha, \beta, \gamma, \delta$ be given (arbitrary) numbers. Find a polynomial of smallest degree $f\colon [a,b] \to \mathbb{R}$ satisfying
i. $f(a) = \alpha$
ii. $f(b) = \beta$
iii. $f'(a) = \gamma$.
iv. $f'(b) = \delta$.

11 Taylor Expansions

Theorem 45 *Let $a, b \in \mathbb{R}$ with $a < b$ and let $n \in \mathbb{N}$. Let $f\colon [a,b] \to \mathbb{R}$ be a function such that $f^{(n-1)}$ is continuous on$[a,b]$ and differentiable on (a,b), i.e. $f^{(n)}(x)\exists$ on (a,b). Then $\exists\ \xi \in (a,b)$ such that*

$$f(b) = f(a)+f'(a)(b-a)+\frac{f''(a)}{2!}(b-a)^2+\cdots+\frac{(b-a)^{(n-1)}}{(n-1)!}f^{(n-1)}(a)+\frac{(b-a)^n}{n!}f^{(n)}(\xi)$$

$$= \sum_{j=0}^{n-1} \frac{(b-a)^j}{j!} f^{(j)}(a) + \frac{(b-a)^n}{n!} f^{(n)}(\xi).$$

Proof. Define a real number K by the identity

$$\frac{(b-a)^n}{n!}K = f(b) - f(a) - \sum_{j=1}^{n-1} \frac{(b-a)^j}{j!} f^{(j)}(a)$$

and define a function φ on $[a,b]$ by

$$\varphi(x) = f(b) - f(x) - \sum_{j=1}^{n-1} \frac{(b-x)^j}{j!} f^{(j)}(x) - \frac{(b-x)^n}{n!}K$$

Then $\varphi(a) = 0$ by the definition of the number K and $\varphi(b) = f(b) - f(b) - 0 - 0 = 0$. Also φ is continuous on $[a,b]$ since the $f^{(j)}$ are continuous for

$0 \le j \le n-1$ and φ is differentiable on (a,b) since $f^{(n)}(x)$ exists on (a,b). Therefore we may apply Rolle's Theorem to φ to infer the existence of a number ξ in (a,b) with $\varphi'(\xi) = 0$.

Now $\varphi'(x) = 0 - f'(x) - \sum_{j=1}^{n-1} \left(-\frac{j(b-x)^{j-1}}{j!} f^{(j)}(x) + \frac{(b-x)^j}{j!} f^{(j+1)}(x) \right) -$ $\frac{-n(b-x)^{(n-1)}}{n!} K$

$$\Rightarrow \varphi'(x) = -f'(x) - [-f'(x) + \frac{(b-x)}{1} f^{(2)}(x) - \frac{(b-x)^1}{1} f^{(2)}(x) + \frac{(b-x)^2}{2!} f^{(3)}(x) \ldots$$

$$-\frac{(b-x)^{(n-2)}}{(n-2)!} f^{(n-1)}(x) + \frac{(b-x)^{(n-1)}}{(n-1)!} f^{(n)}(x)] + \frac{(b-x)^{(n-1)}}{(n-1)!} K$$

Therefore,

$$0 = \varphi'(\xi) = -\frac{(b-\xi)^{(n-1)}}{(n-1)!} f^{(n)}(\xi) + \frac{(b-\xi)^{(n-1)}}{(n-1)!} K$$

and so $K = f^{(n)}(\xi)$. Putting this expression back into the defining expression for K gives the Taylor Expansion and completes the proof. □

If $b < a$, then the expansion still exists with $\xi \in (b,a)$.
If $b = a + h$ then the expansion may be written

$$f(a+h) = f(a) + \sum_{j=1}^{n-1} \frac{f^{(j)}(a)}{j!} h^j + \frac{h^n}{n!} f^{(n)}(a + \theta h)$$

for some θ (depending on a, h, f) with $|\theta| < 1$.

The reader will note that Rolle's Theorem is used in the proof of Taylor's Theorem. Some regard the latter as the most important theorem in analysis because of its use in obtaining approximations and deriving power series (see later). However it appears that Rolle's Theorem is more fundamental since from it one may prove:

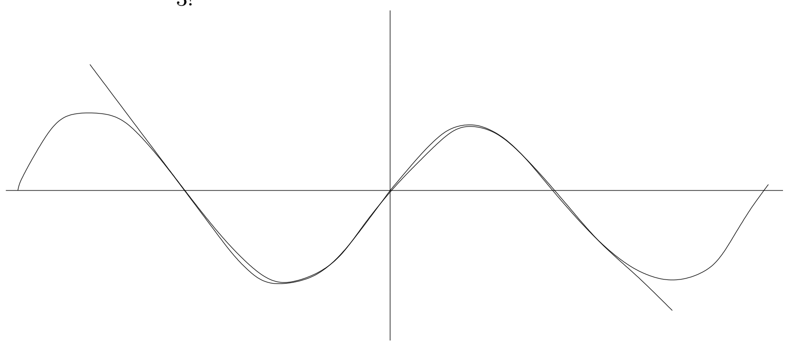
i. The Mean Value Theorem
ii. Taylor's Theorem

iii. The principle of identical derivatives
iv. The increasing value principle for differentiable functions
v. The fundamental theorem of integral calculus

Traditionally, Taylor's Theorem has been used for functions with an infinite number of derivatives on the whole of $\mathbb{R}$; the case $a = 0, b = x$ being given special prominence. Then

$$f(x) = f(0) + \sum_{j=1}^{n-1} \frac{f^{(j)}(0)}{j!} x^j + \frac{x^n}{n!} f^{(n)}(\theta x).$$

When x is close to zero, ($|x| << 1$, meaning x much smaller than 1) then x^n is smaller than the lower powers of x and the polynomial of degree $n-1$ given by the Taylor expansion will be a good approximation to $f(x)$. For example, $\sin x \approx x - \frac{x^3}{3!}$.



To get a good approximation to, say, 10 periodic sections of sin one needs a Taylor expansion of degree about 40!! Easy to write down but hopeless from a computational point of view.

The point of all this discussion is don't overrate Taylor expansions (most scientists do); the temptation is to start using one for $|x| << 1$ and then assume it holds $\forall x \in \mathbb{R}$!.
e.g.
1.

$$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + ... \qquad -1 < x \leq 1$$

2.

$$e^x = 1 + \frac{x}{1} + \frac{x^2}{2!} + ... \qquad \forall x \in \mathbb{R}$$

3.

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - ... \qquad \forall x \in \mathbb{R}$$

4.

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - ... \qquad \forall x \in \mathbb{R}$$

5.

$$\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{7x^7}{315} + ... \qquad |x| < \frac{\pi}{2}$$

6.

$$\text{cosec } x = \frac{1}{x} + \frac{x}{6} + \frac{7}{360}x^3 + \frac{31}{15120}x^5 + ... \qquad \forall 0 < |x| < \pi$$

7.

$$\sec x = 1 + \frac{x^2}{2} + \frac{5x^4}{24} + 61x^6 720 + ... \qquad |x| < \frac{\pi}{2}$$

8.

$$\cot x = \frac{1}{x} - \frac{x}{3} - \frac{x^3}{45} - \frac{2x^5}{945} ... \qquad |x| < \pi$$

9.

$$\arcsin x = x + \frac{x^3}{2 \cdot 3} + \frac{1 \cdot 3x^5}{2 \cdot 4 \cdot 5} + \frac{1 \cdot 3 \cdot 5x^7}{2 \cdot 4 \cdot 6 \cdot 7} \qquad |x| < 1$$

10.

$$\arctan x = \begin{cases} x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} ... & |x| \leq 1 \\ \frac{\pi}{2} - \frac{1}{x} + \frac{1}{3x^2} - \frac{1}{5x^5} ... & |x| > 1 \end{cases}$$

11.

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + ... \qquad x \in \mathbb{R}$$

12.

$$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + ... \qquad x \in \mathbb{R}$$

13.

$$\tanh x = x - \frac{x^3}{3} + \frac{2x^5}{15} - \frac{17x^7}{315} + ... \qquad |x| < \frac{\pi}{2}$$

14.

$$\text{csch } x = \frac{1}{x} - \frac{x}{6} + \frac{7x^3}{360} - \frac{31x^5}{15120} ... \qquad 0 < |x| < \pi$$

15.

$$\text{sech } x = 1 - \frac{x^2}{2} + \frac{5x^4}{24} - \frac{6x^6}{720} + ... \qquad |x| < \frac{\pi}{2}$$

16.

$$\coth x = \frac{1}{x} - \frac{x}{3} - \frac{x^3}{45} + \frac{2x^5}{945} ... \qquad 0 < |x| < \pi.$$

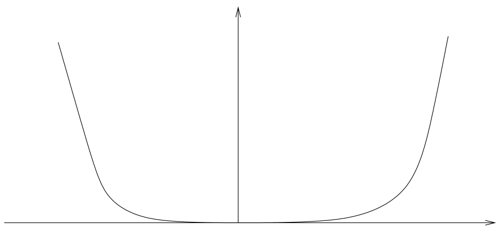
Theorem 46 Sufficient conditions for maxima and minima
Let $f \in C^{(2n)}[a,b]$, i.e. $f^{(2n)}(x)$ is continuous on $[a,b]$. If $\exists \xi \in (a,b)$ such that $f^{(j)}(\xi) = 0$ for $1 \le j \le 2n-1$ then
i. If $f^{(2n)}(\xi) < 0$ then f will have a local maxima at ξ.
ii. If $f^{(2n)}(\xi) > 0$ then f will have a local minima at ξ.

Proof. ii. Note that if g is a continuous function on $[a,b]$ and $a < \xi < b$ and $g(\xi) > 0$ then $\exists \delta > 0$ such that $g(x) > 0$ when $\xi - \delta < x < \xi + \delta$. We will apply this easily proved result to $f^{(2n)}$. The condition given on f ensure that the Taylor expansion with remainder of order $2n$ exists as follows about ξ:

$$f(\xi + h) = f(\xi) + 0 + 0 + ... + 0 + \frac{h^{2n}}{(2n)!} f^{(2n)}(\xi + \theta h)$$

for some $\theta = \theta_h$ with $0 < \theta < 1$. Let $|h| < \delta$. Then $|(\xi + \theta_h) - \xi| = |\theta_h| < \delta$ so that $f^{(2n)}(\xi + h) > 0$. Hence $f(\xi + h) > f(\xi)$ for $0 < |h| < \delta$ and therefore f has a local minimum at ξ. □

e.g.
i. $f(x) = x^4 + x^8$

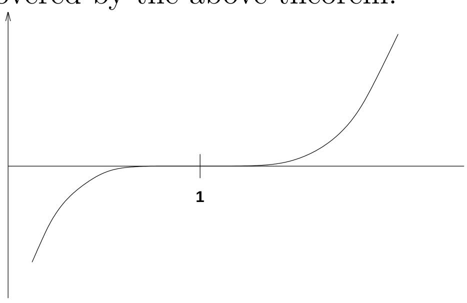


Then $f^{(1)}(0) = f^{(2)}(0) = f^{(3)}(0) = 0$ and $f^{(4)}(0) = 4 > 0$.
Thus f has a local minimum at 0 by part ii.

ii. $f(x) = (x-1)^7$

$$f^{(1)}(1) = 0 = ... = f^{(6)}(1) = 0 \text{ and } f^{(7)}(1) \neq 0$$

and this f is **not** covered by the above theorem.

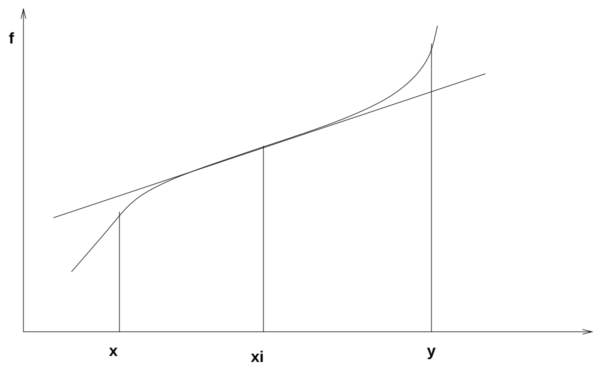


Definition Let f have a continuous first derivative on $[a,b]$. We say f has a (not necessarily horizontal) **point of inflection** at ξ if the graph of f crosses the tangent at $(\xi, f(\xi))$, i.e. if

$$d_1 = f(x) - f(\xi) - (x-\xi)f'(\xi)$$

$$d_2 = f(y) - f(\xi) - (y-\xi)f'(\xi)$$

for some $\delta > 0$ and $\xi - \delta < x < \xi < y < \xi + \delta$ have opposite signs.



Theorem 47 *Let* $f\colon [a,b] \to \mathbb{R}$ *have a continuous* $(2n+1)^{th}$ *derivative on* $[a,b]$. *Let* $\xi \in (a,b)$ *be such that* $f^{(j)}(\xi) = 0$ *for* $2 \leq j \leq 2n$ *and let* $f^{(2n+1)}(\xi) \neq 0$. *Then* $(\xi, f(\xi))$ *is a point of inflection.*

Proof.

$$f(\xi+h) = f(\xi) + hf'(\xi) + 0 + ... + 0 + \frac{h^{2n+1}}{(2n+1)!}f^{(2n+1)}(\xi+\theta h)$$

for some $\theta = \theta_h$ with $0 < \theta < 1$, by Taylor's Theorem.
Because $f^{(2n+1)}$ is continuous and $f^{(2n+1)}(\xi) \neq 0, \exists \delta > 0$ such that $f^{(2n+1)}(\xi+\theta h) \neq 0$ for $|h| < \delta$ and does not change sign.

If $h_2 > 0$ then $\frac{h_2^{2n+1}}{(2n+1)!} > 0$
If $h_1 < 0$ then $\frac{h_1^{2n+1}}{(2n+1)!} < 0$.

Therefore,

$$d_1 = f(\xi+h_1) - f(\xi) - h_1 f'(\xi) = \frac{h_1^{2n+1}}{(2n+1)!}f^{(2n+1)}(\xi+\theta h_1)$$

and

$$d_2 = f(\xi+h_2) - f(\xi) - h_2 f'(\xi) = \frac{h_2^{2n+1}}{(2n+1)!}f^{(2n+1)}(\xi+\theta h_2)$$

have opposite signs. □

Exercises
i. Expand $f(x) = x^3 - x^4$ in a Taylor series with remainder of order 4 and then with remainder of order 5 about $x = 0$ and then about $x = 1$. Interpret graphically and show how the expansions of order 2 and 3 about the two points approximate the function.

ii. Use the Taylor expansion of order 4 for $\sin x$ to evaluate

$$\lim_{h \to 0} \frac{\sin x - x + x^3}{x^3 + x^7}$$

iii. Investigate the maxima, minima, and points of inflection of
a. $f(x) = x^4 - x^5$ and
b. $g(x) = x^5 + x + 1$.

12 Applications of Taylor's Theorem

12.1 Big Oh Notation

We say $f = 0(g)$ as $x \to a$ if $\exists \delta < 0$ and an $M > 0$ such that $|f(x)| \leq M|g(x)|$ when $|x - a| < \delta$.
We say $f = 0(g)$ as $x \to \infty$ if $\exists \Delta > 0$ and an $M > 0$ such that $|f(x) \leq M|g(x)|$ when $x > \Delta$.

For example,
i. $x = 0(2x)$ as $x \to 0$ and as $x \to \infty$
ii. $x = 0(x^2)$ as $x \to \infty$ but not as $x \to 0$.
iii. $x^2 = 0(x)$ as $x \to 0$ but not as $x \to \infty$.

Write $f < g$ if $f = 0(g)$. Then $f < f$, $f < g$, $g < h \Rightarrow f < h$ but $f < g$ and $g < f$ are possible with $f \neq g$ [consider x and $x + \frac{1}{x}$ as $x \to \infty$].

Let f be such that its nth derivative exists and is continuous on $[-a, a]$. Then $|f^{(n)}(x)| \leq M$ for some M and all $x \in [-a, a]$.

Notation: by $f = g + 0(k)$ where f, g, k are functions we mean $f - g = 0(k)$.

By Taylor's Theorem,

$$f(0+h)-f(0)-f'(0)h-...-\frac{f^{(n-1)}(0)}{(n-a)!}h^{n-1}=\frac{f^{(n)}(\theta h)}{n!}h^n$$

Since $|\frac{f^{(n)}(\theta h)}{n!}| \leq \frac{M}{n!}$ we may write

$$f(h)=f(0)+f'(0)h+...+\frac{f^{(n-1)}(0)}{(n-1)!}h^{n-1}+0(h^n).$$

Exercise

$$\lim_{h\to 0}\frac{0(h^n)}{h^{n-1}}=0 \text{ for } n=1,2,3,....$$

The big Oh notation is an efficient way of extracting the main property of the remainder, especially in applications to numerical analysis.

12.2 Indeterminate Forms

Theorem 48 *Suppose that f and g and their derivatives up to order $n-1$ are continuous for $a \leq x \leq a+h$ and that for $a < x < a+h, f^{(n)}(x)$ exists and that $g^{(n)}(x)$ exists and is not equal to zero. Then if*

$$f(a)=f'(a)=...=f^{(n-1)}(a)=0$$

and

$$g(a)=g'(a)=...=g^{(n-1)}(a)=0$$

and

$$\lim_{x\to a+}\frac{f^{(n)}(x)}{g^{(n)}(x)}=l,$$

then $\lim_{x\to a+}\frac{f(x)}{g(x)}=l.$

Proof. Let $\phi(x)=f(x)-Kg(x)$.
By Taylor's Theorem, $\phi(a+h)=\frac{h^n}{n!}\phi^{(n)}(a+\theta h)$.

Choose K so that $\phi(a+h)=0$. Note that $\phi^{(n)}(x)=f^{(n)}(x)-Kg^{(n)}(x)$ so $K=\frac{f(a+h)}{g(a+h)}=\frac{f^{(n)}(a+h)}{g^{(n)}(a+h)}$. Thus,

$$\lim_{x\to a+}\frac{f^{(n)}(x)}{g^{(n)}(x)}=\lim_{h\to 0+}\frac{f(a+h)}{g(a+h)}.$$

□

e.g.
i. $f(x)=\sin x,\ g(x)=x$ then $f(0)=0=g(0)$.
But $\frac{f'(x)}{g'(x)}=\frac{\cos x}{1}$ and $\lim_{x\to 0}\cos x=1$.
Thus $\lim_{x\to 0}\dfrac{\sin x}{x}=1$.

ii. $f(x)=\sin x-x, g(x)=x^3$.
Then $f(0)=f'(0)=f''(0)=0$ and $g(0)=g'(0)=g''(0)=0$.
But $\frac{f'''(x)}{g'''(x)}=-\frac{\cos x}{6}\to-\frac{1}{6}$ as $x\to 0$.
Hence $\lim_{x\to 0}\dfrac{\sin x-x}{x^3}=-\dfrac{1}{6}$.

Using Big Oh:

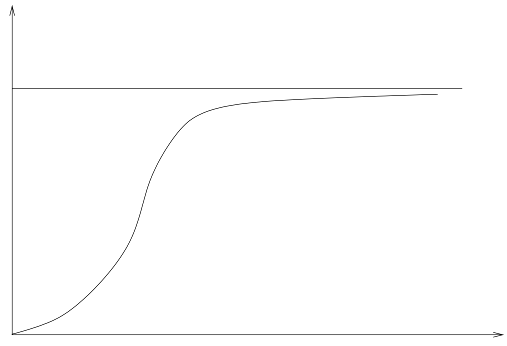
$$\sin x=x-\frac{x^3}{6}+0(x^5).$$

Hence

$$\frac{\sin x-x}{x^3}=\frac{-\frac{x^3}{6}+0(x^5)}{x^3}=-\frac{1}{6}+\frac{0(x^5)}{x^3}\text{ and }\lim_{x\to 0}\frac{0(x^5)}{x^3}=0.$$

e.g. Let

$$f(x)=\begin{cases} e^{-\frac{1}{x^2}} & x>0\\ 0 & x\le 0\end{cases}$$



Then f is C^∞ on $\mathbb{R}$ and $f^{(n)}(0) = 0\ \forall n \in \mathbb{N}$:

$$f'_+(0) = \lim_{h\to 0+} \frac{f(h) - f(0)}{h} = \lim_{h\to 0} \frac{e^{-\frac{1}{h^2}}}{h}$$

$$= \lim_{h\to 0} \frac{\frac{1}{h}}{e^{\frac{1}{h^2}}}$$

$$= \lim_{h\to 0} \frac{(\frac{1}{h})'}{(e^{\frac{1}{h^2}})'} \quad \text{if this limit exists}$$

$$= \lim_{h\to 0} \frac{-\frac{1}{h^2}}{-\frac{2e^{\frac{1}{h^2}}}{h^3}} = 0.$$

$$f'_-(0) = \lim_{h\to 0} \frac{f(h) - f(0)}{h} = 0.$$

Hence $f'(0) = 0$.
If $x > 0$ then $f'(x) = (e^{-\frac{1}{x^2}})' = \frac{2}{x^3} e^{-\frac{1}{x^2}}$.
If $x < 0$ then $f'(x) = 0 = f''(x) = f^{(n)}(x)\ \forall n$.

To complete the example we need to know that if for a given $m \in \mathbb{N}$

$$g(x) = \begin{cases} \frac{e^{-\frac{1}{x^2}}}{x^{m-1}} & x > 0 \\ 0 & x \le 0 \end{cases}$$

then $g'_+(0)$ exists and is zero. Now

$$g'_+(0) = \lim_{h\to 0+} \frac{\frac{1}{h^m}}{e^{\frac{1}{h^2}}}$$

$$= c_1 \lim_{h\to 0} \frac{\frac{h^3}{h^{m+1}}}{e^{\frac{1}{h^2}}} = c_2 \lim_{h\to 0} \frac{\frac{h^6}{h^{m+2}}}{e^{\frac{1}{h^2}}}$$

$= ... = c_j \lim_{h\to 0} \frac{\frac{h^3}{h^{m+1}}}{e^{\frac{1}{h^2}}}$ and when $3j > m + j \Leftrightarrow 2j > m$,
$= 0.$
This completes the calculation. Note that f always has a finite Taylor series but $f(x) = R_n(x)$ the order n remainder $\forall n \in \mathbb{N}$ and then

$$0 < f(x) = \sum_0^\infty \frac{f^{(n)}(x)}{n!} x^n$$

is false even though f is C^∞ or smooth. We say f is not **real analytic**.

12.3 Order of Difference Quotient

Note that $\frac{0(h^n)}{h} = 0(h^{n-1})$ and so on. If f has a sufficient number of continuous derivatives we may write

$$f(a+h) - f(a) = f'(a)h = \frac{1}{2!} f''(a)h^2 + 0(h^3).$$

Thus $\frac{f(a+h)-f(a)}{h} = f'(a) + 0(h^1)$ when $h \to 0$.
We say that the usual Newton quotient gives a first order approximation to $f'(a)$ and we call the 0(h) term the local truncation error. It is interesting to see if the alternative difference quotient is any better in this sense.

$$f(a+h) = f(a) + f'(a)h + f''(a)\frac{h^2}{2!} + 0(h^3) \text{ and}$$

$$f(a-h) = f(a) - f'(a)h + f''(a)\frac{h^2}{2} + 0(h^3).$$

Subtract:

$$\frac{f(a+h) - f(a-h)}{2h} = f'(a) + 0(h^2), \text{ i.e. second order.}$$

Now write

$$f(a+h) = f(a) + f'(a)h + f''(a)\frac{h^2}{2!} + f'''(a)\frac{h^3}{3!} + 0(h^4)$$

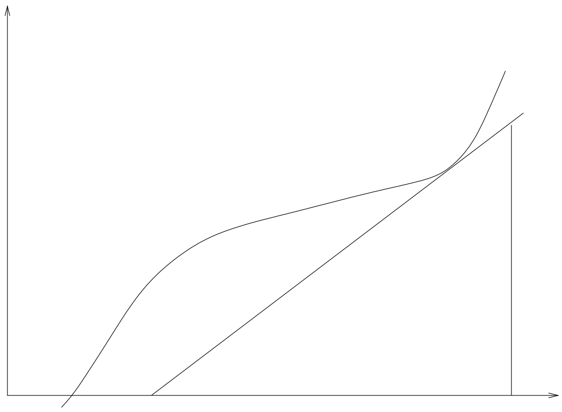
$$f(a-h) = f(a) - f'(a)h + f''(a)\frac{h^2}{2!} - f'''(a)\frac{h^3}{3!} + 0(h^4).$$

Add:
$$\frac{f(a+h)-2f(a)+f(a-h)}{h^2} = f''(a)+0(h^2).$$
Note that this implies the LHS $\to f''(a)$ when $h \to 0$.

Exercise Is it possible to find a 3^{rd} order quotient for the 2^{nd} derivative?

12.4 Newton's Method for finding a root

A claim was made earlier that fast methods exist for finding roots for suitably differentiable functions.

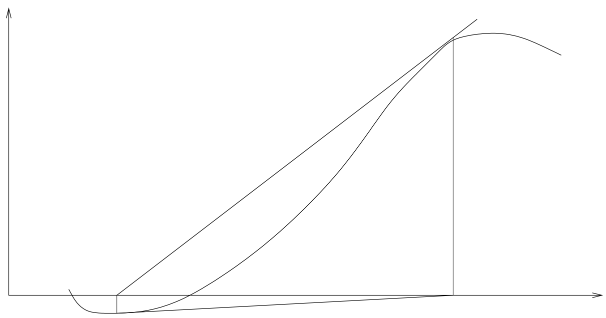


From the given triangle, $r_{n+1} = r_n - \frac{f(f_n)}{f'(r_n)}$. (1)
Experiment shows that provided the value of r_0 is reasonably close to a root and $f'(x)$ is not near zero in the neighbourhood of r then $r_n \to r$ rapidly. Taylor's expansion with remainder of order 2 may be used to quantify this convergence.

$$f(r) = f(r_n) + f'(r_n)(r-r_n) + \frac{1}{2}f''(\xi)(r-r_n)^2$$

where ξ lies between r and r_n and will depend on $n \in \mathbb{N}$.
Thus $f(r_n) = -f'(r_n)(r-r_n) - \frac{1}{2}f''(\xi)(r-r_n)^2$. Substitute this value for $f(r_n)$ in (1)
$$r_{n+1} = r_n + (r-r_n) + \frac{f''(\xi)}{2f'(r_n)}(r-r_n)^2.$$
Thus $r_{n+1} - r = M_n(r-r_n)^2$ (2)
where $M_n = \frac{f''(\xi)}{2f'(r_n)}$. Therefore r_{n+1} will be a better approximation to r than

r_n if $|M_n(r - r_n)| < 1$. It is interesting to note that even if r_n were close to r but M_n were large the method might not converge:



For the method to converge we need $|M_n| < 1$, i.e. $f''(x)$ not large and $f'(x)$ bounded away from zero near $x = r$.

13 Integration I

We will develop integration theory for bounded real valued functions on closed bounded intervals in $\mathbb{R}$. The definition of the integral given below is one of the easiest to work with but one you will probably not be familiar with. Other equivalent definitions will be given later. In practice this theory, and its extension to functions on $\mathbb{R}^n$ for $n > 1$, is all that one ever needs in applications. For theoretical purposes the Lebesgue theory of integration is much better: it is usually studied in the fourth or fifth year.

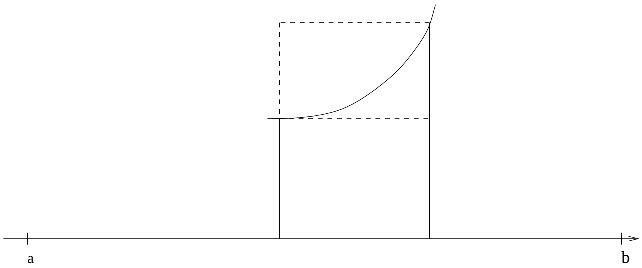
Definition A **partition** P of a closed interval $[a, b]$ is a sequence $(x_0, ..., x_n)$ with $a = x_0 \leq x_1 \leq ... \leq x_n = b$.

The partition P divides the interval $[a, b]$ into n subintervals $[x_{i-1}, x_i] = I_i$. We call this latter set the i^{th} subinterval of the partition P and write $I_i \in P$. Let $f\colon [a, b] \to \mathbb{R}$ be a **bounded** function and let P be partition of $[a, b]$. Let $S \in P$ and let

$$m_s(f) = inf\{f(x) : x \in S\},$$

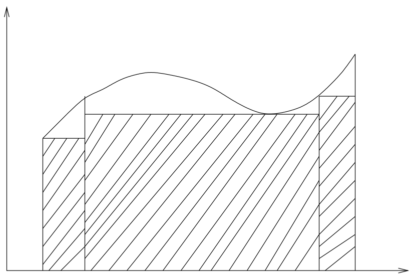
$$M_s(f) = sup\{f(x) : x \in S\}.$$

If $S = I_i$ we write $m_i = m_S(f)$ and $M_i = M_S(f)$.
Write $\Delta x_i = x_i - x_{i-1} = l(I_i)$, the length of I_i.



The **lower sum** of f for P is given by $L(f,P) = \sum_{i=1}^{n} m_i \Delta x_i = \sum_{i=1}^{n} m_{I_i}(f) \Delta x_i$.

The **upper sum** of f for P is given by $U(f,P) = \sum_{i=1}^{n} M_i \Delta x_i = \sum_{i=1}^{n} M_{I_i}(f) \Delta x_i$.



For example, the value of the lower sum for the function in the above diagram is the sum of the areas of the three shaded rectangles.

Since $m_1 \leq M_i$ for $1 \leq i \leq n$ it follows that

$$L(f,P) \leq U(f,P) \qquad (1)$$

for all bounded f and for all partitions P. We approximate the area under the graph by choosing partitions P which become progressively finer.

Definition We say a partition P' **refines** a partition P if every point in P is a point in P'.

For example if $P' = (0,1,2,3,4)$ and $P = (0,2,4)$ are partitions on $[0,\ 4]$ then P' refines P.

Theorem 49 *If P' refines P then $L(f,P) \leq L(f,P') \leq U(f,P') \leq U(f,P)$.*

Proof. The middle inequality is an immediate consequence of equation (1) above. We will prove only the left hand inequality below.
Each subinterval I of P is divided into several (or one) subintervals $I_i, ..., I_n$ of P' and $l(I) = l(I_1) + l(I_2) + ... + l(I_n)$.

Now $m_I(f) \leq m_{I_j}(f)$ since the values of $f(x)$ for $x \in I$ include all values of $f(x)$ for $x \in I_j$ for $1 \leq j \leq n$. Thus

$$m_I(f) \cdot l(I) = m_I(f)l(I_1) + ...m_I(f)l(I_n) \leq m_{I_1}(f)l(I_1) + ... + m_{I_n}(f)l(I_n).$$

The sum for all $I \in P$ of the terms on the LHS is $L(f,P)$, while the sum of all the terms on the RHS is $L(f,P')$. Thus $L(f,P) \leq L(f,P')$. □

Corollary If P and P' are any two partitions then $L(f,P) \leq U(f,P')$.

Proof. Let P'' be a partition which refines both P and P'. (For example it may be the partition whose set of points consists of the union of the sets of points of P and P'). Then $L(f,P) \leq L(f,P'') \leq U(f,P'') \leq Uf(,P')$. □

It follows from this corollary that

$$sup_P L(f,P) \leq inf_P U(f,P) \qquad (2)$$

where the inf and sup are taken over all partitions P, i.e. over all upper and lower sums respectively.

Definition A function $f\colon [a,b] \to \mathbb{R}$ is **Riemann Integrable** if there is equality in the above inequality.

Later we will see that all continuous, all bounded monotonic and all bounded piecewise continuous functions are Riemann Integrable and show that the Riemann integral has the same value as the definite integral calculated using the fundamental theorem of integral calculus.

It is convenient to develop first a useful criteria for integrability:

Theorem 50 *A bounded function $f\colon [a,b] \to \mathbb{R}$ is Riemann integrable if and only if for every $\varepsilon > 0$ $\exists$ a partition P of $[a,b]$ such that*

$$0 \leq U(f,P) - L(f,P) < \varepsilon.$$

Proof. Assume the given condition. Given $n \in \mathbb{N}$, there is a partition P_n such that

$$inf_P U(f,P) \leq U(f,P_n) < L(f,P_n) + \frac{1}{n} \leq sup_P L(f,P) + \frac{1}{n}.$$

If $n \to \infty$ we obtain $inf_P U(f,P) \leq sup_P L(f,P) \leq inf_P U(f,P)$ where the last inequality follows from equation (2) above. Therefore, f is Riemann integrable.

Conversely, if f is Riemann integrable then

$$inf_P U(f,P) = sup_P L(f,P) = \lambda, \text{ say.}$$

Given $\varepsilon > 0$, there is a partition P'' such that

$$0 \leq \lambda - L(f,P'') < \frac{\varepsilon}{2}$$

and a partition P' such that

$$0 \leq U(f,P') - \lambda < \frac{\varepsilon}{2}.$$

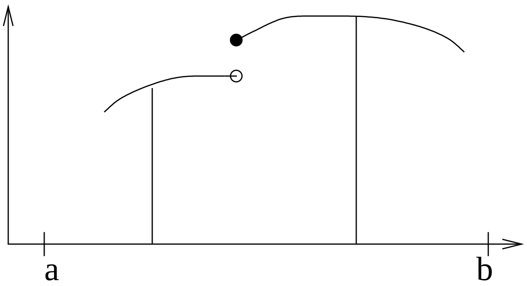
If P refines P' and P'' then

$$U(f,P) - L(f,P) \leq U(f,P') - L(f,P'') < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

□

Theorem 51 *If $f\colon [a,b] \to \mathbb{R}$ is monotonically increasing and bounded then f is Riemann integrable.*

Proof. Given$\varepsilon > 0$ choose $n \in \mathbb{N}$ so that $\frac{(b-a)(f(b)-f(a))}{n} < \varepsilon$.
Let P_n be the partition formed by dividing $[a,b]$ into n subintervals of equal width $\frac{b-a}{n}$. Then $\Delta x_i = \frac{b-a}{n}$ for $1 \leq i \leq n$.



Since f is monotonically increasing $m_j = f(x_{j-1}) and M_j = f(x_j), 1 \leq j \leq n$.
Then

$$U(f, P_n) - L(f, P_n) = \sum_{j=1}^{n} f(x_j)\Delta x_j - \sum_{j=1}^{n} f(x_{j-1})\Delta x_j$$

$$= (\sum_{j=1}^{n} f(x_j) - f(x_{j-1}))(\frac{b-a}{n})$$

$$= (f(b) - f(a))\frac{(b-a)}{n} < \varepsilon.$$

Therefore f is Riemann integrable. □

14 Integration II

If f is integrable on $[a, b]$ then the common value

$$inf_P U(f, P) = sup_P L(f, P)$$

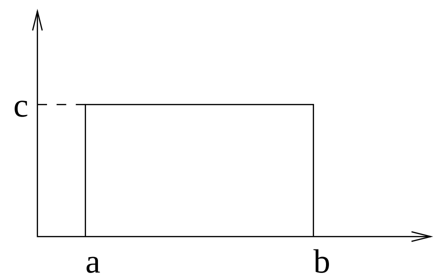
is written $\int_a^b f$ or $\int_a^b f(x)\,dx$ or $Int(f, a, b)$.

Theorem 52 *Let $f, g \colon [a, b] \to \mathbb{R}$ be (Riemann) Integrable and let $c \in \mathbb{R}$. Then*
i. The constant function h_c is integrable and $\int_a^b h_c = c(b - a)$
ii. The function cf is integrable and $\int_a^b cf = c\int_a^b f$
iii. $f + g$ is integrable and $\int (f + g) = \int f + \int g$
iv. If $f \geq 0$ then $\int f \geq 0$
v. If $f \geq g$ then $\int f \geq \int g$
vi. $|f|$ is integrable and $\int |f| \geq |\int f|$

Proof.
i. Since for all intervals $I \subset [a,b], m_I(h_c) = M_I(h_c) = c$,

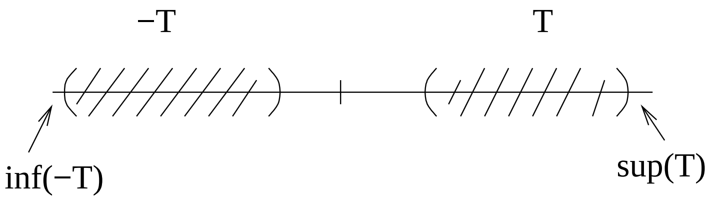
$$L(h_c, P) = \sum c\Delta x_i = c(b-a) \text{ and } U(h_c, P) = \sum c\Delta x_i = c(b-a)$$

and so the value of the integral is $c(b-a)$.



ii. If $c = 0$ then $(cf)(x) = 0 = h_0(x)$ and the result is immediate.
Let $c = -1$. Note that if $T \subset \mathbb{R}$ is a non-empty subset

$$-sup(T) = inf(-T) \qquad (1).$$



Hence $\forall I \subset [a,b]$,

$$-M_I(f) - m_I(-f)$$

and so for all partitions P,

$$-U(f,P) = L(-f,P) \qquad (2)$$

Hence $\quad sup_P L(-f,P) = sup_P(-U(f,P))$

$$= -inf_P U(f,P) \quad \text{by (1)}$$

$$= -sup_P L(f,P) \text{ since f is integrable}$$

$= inf_P(-L(f,P))$

$$= inf_P U(-f,P) \text{ by (2) applied to } -f.$$

Thus $-f$ is integrable and, by the second equality,

$$\int -f = -\int f.$$

If $c < 0$, note that $sup(cT) = c \cdot sup(T)$ and

$$inf(cT) = c \cdot inf(T) \quad \text{for } T \subset \mathbb{R}.$$

Thus $L(c.f,P) = cL(f,P)$ and $U(cf,P) = cU(f,P)$. The result follows easily from these equalities.

If $c < 0$ then $cf = (-c)(-1)f$ and the result follows by the above cases since $-c > 0$.

iii. First note that if $T, T' \subset \mathbb{R}$ are bounded subsets then

$$inf\ T + inf\ T' \leq inf(T+T');$$

To see this given $\varepsilon > 0$ there is a $t \in T$ and a $t' \in T'$ such that

$$t + t' < inf(T+T') + \varepsilon.$$

But

$$inf\ T \leq t \text{ and } inf\ T' \leq t'.$$

Thus

$$inf\ T + inf\ T' < inf(T+T') + \varepsilon \quad \forall \varepsilon > 0.$$

Thus

$$inf\ T + inf\ T' \leq inf(T+T'). \qquad (3)$$

Similarly, $sup(T+T') \leq sup\ T + sup\ T'$ (in fact, $inf\ T + inf\ T' = inf(T+T')$ and ditto for sup). From equation (3) $m_S(f) + m_S(g) \leq m_S(f+g)$ and from equation (4) $M_S(f+g) \leq M_S(f) + M_S(g)$.
Hence

$$L(f,P) + L(g,P) \leq L(f+g,P) \leq U(f+g,P) \leq U(f,P) + U(g,P).$$

Given $\varepsilon > 0$ there are partitions P', P'', P''', P'''' such that

$$\int f - \varepsilon < L(f, P')$$

$$\int g - \varepsilon < L(g, P'')$$

$$U(f, P''') < \int f + \varepsilon$$

$$U(g, P'''') < \int g + \varepsilon.$$

Let P be a common refinement of these 4 dashed partitions. Then, by equation (4),

$$\int f + \int g - 2\varepsilon \leq L(f+g, P) \leq U(f+g, P) \leq \int f + \int g + 2\varepsilon.$$

Hence $f+g$ is integrable and $\int(f+g) = \int f + \int g$.

iv. If $f \geq 0$ then $m_S(f) \geq 0 \,\forall S \in P$ and so $L(f, P) \geq 0$. Thus $\int f \geq 0$.

v. If $f \geq g$ then $f - g \geq 0$ and so, by iv., $\int(f-g) \geq 0$.
But, by ii. and iii., $\int(f-g) = \int f - \int g$.
Thus $\int f - \int g \geq 0 \Rightarrow \int f \geq \int g$.

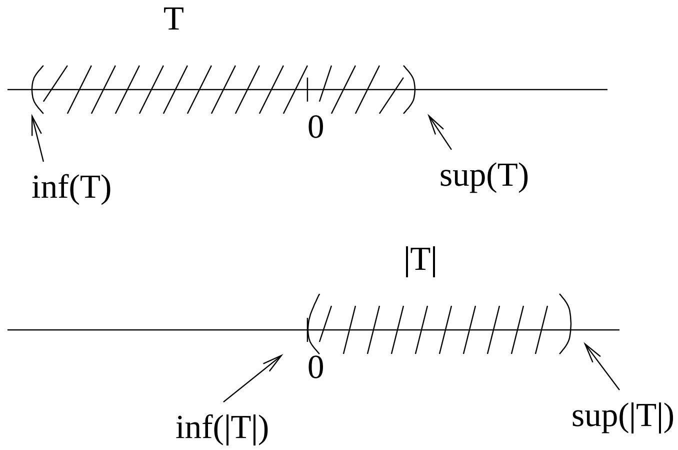
vi. Let $S \subset [a,b]$ be a subinterval. If $f \geq 0$ on S then $|f| = f$ and so

$$M_S(|f|) - m_S(|f|) = M_S(f) - m_S(f)$$

from which inequality it follows that $|f|$ is integrabl.e If $f \leq 0$ on S then $|f| = -f$ and so

$$\begin{aligned} M_S(|f|) - m_S(|f|) &= M_S(-f) - m_S(-f) \\ &= -m_S(f) - -M_S(f) \\ &= M_S(f) - m_S(f) \end{aligned}$$

and we are done. In general, f might be both negative and positive on S:



If $T \subset \mathbb{R}$ is bounded then

$$sup|T| - inf|T| \leq supT - inf\ T$$

The only case to offer any real difficulties is when $inf\ T < 0 < supT$ and $sup|T| = -inf|T|$. Then necessarily $0 \leq inf|T|$ and so

$$sup|T| - inf|T| \leq -inf|T| \leq supT - inf\ T.$$

Applying this inequality,

$$M_S(|f|) - m_S(|f|) \leq M_S(f) - m_S(f)$$

for all f and all $S \subset I$. The integrability of $|f|$ follows from this inequality and the integrability of f in itself.
Finally, since $f \leq |f|$ and $-f \leq |f|$ by ii. and v.

$$\int f \leq \int |f| \text{ and } -\int f \leq \int |f|.$$

Thus $\quad |\int f| \leq \int |f|$. This completes the proof of the theorem. □

Note The work involved in proving this standard theorem is difficult to justify. On the surface it looks like a machine for showing functions and families of functions are integrable. In practice all one really usually needs are the results:
i. monotonically increasing or decreasing functions are integrable
ii. continuous functions are integrable

iii. if a function is integrable on two adjacent intervals it is integrable on their union
iv. if a function is integrable on an interval it is integrable on any subinterval.

We will prove ii - iv in a future lecture. Surprisingly it is very difficult to show that the product of two integrable functions is integrable - this is best done after a theorem which shows exactly when a function is or is not integrable has been proven, namely:
" A function is integrable if and only if the set of points at which it is not continuous has measure zero, i.e. is a very small set". This is proved in a second analysis course.

Exercise An indefinite integral for $\int_0^b \frac{\sin x}{x}\,dx$ does not exist and you do not wish to use a numerical method because the upper limit b is to be left as a parameter. Use Taylor's theorem and Thm 52 v. to find functions f and g such that

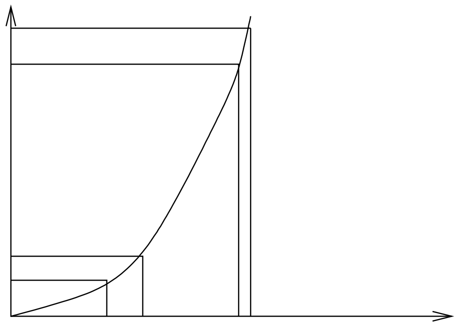
$$f(b) \leq \int_0^b \frac{\sin x}{x}\,dx \leq g(b) \quad \text{for} \quad 0 \leq b \leq \frac{\pi}{2}.$$

15 Uniform Continuity

In the definition of continuity

$$\forall \varepsilon > 0, \exists \delta_\varepsilon > 0 \text{ such that } |x-a| < \delta_\varepsilon \Rightarrow |f(x) - f(a)| < \varepsilon$$

frequently, the fact that δ_ε depends on a is suppressed. Even with a function as smooth as $x \to x^2$ it is impossible to make a uniform choice and as a increases, $\delta_\varepsilon(a)$, for a given$\varepsilon > 0$, must become smaller to satisfy the given inequality.



In the next lecture we will show that if the continuous function is restricted to a bounded closed interval then the δ_ε can be chosen uniformly for every point in the interval.

Definition We say $f\colon S \to \mathbb{R}$ is uniformly continuous on S if $\forall \varepsilon > 0, \exists \delta_\varepsilon > 0$ such that $\forall x, y \in S$ if

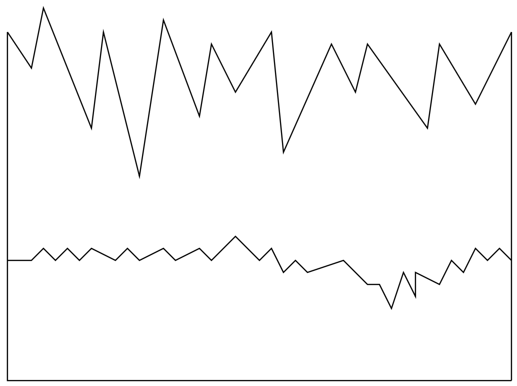
$$|x - y| < \delta_\varepsilon \text{ then } |f(x) - f(y)| < \varepsilon.$$

e.g. Suppose $f\colon [a, b] \to \mathbb{R}$ satisfies $|f(x) - f(y)| \leq M|x - y| \forall x, y \in [a, b]$. Then f is uniformly continuous on $[a, b]$. To see this, choose $\delta_\varepsilon = \frac{\varepsilon}{M}$.

This is a type of bounded slope condition with M being an upper bound on the slopes

e.g. $|\sin x - \sin y| \leq |x - y|$.

However, a function could have no slope at many points and still satisfy such an inequality.



e.g. $f\colon [a,\infty) \to \mathbb{R}, \quad a > 0$
$x \to \sqrt{x}$.
Then f is uniformly continuous.
Given $\varepsilon > 0$, let $\delta_\varepsilon = \sqrt{a}\varepsilon$. Then if $x \neq y$ and $\geq a$

$$|x^{\frac{1}{2}} - y^{\frac{1}{2}}| = \frac{|x-y|}{x^{\frac{1}{2}} + y^{\frac{1}{2}}} < \frac{|x-y|}{x^{\frac{1}{2}}} \leq \frac{|x-y|}{a^{\frac{1}{2}}} < \frac{\sqrt{a}\varepsilon}{\sqrt{a}} = \varepsilon$$

if $|x-y| < \delta_\varepsilon$.

Theorem 53 *Let (a_n) be a bounded sequence. Then there exists a subsequence (a_{n_j}) and real number α such that $a_{n_j} \to \alpha$.*

Proof. For each $n \in \mathbb{N}$, let $b_n = inf\{a_n, a_{n+1}, ...\}$. Then there is a real number b such that $b_n \uparrow b$ since the sequence (b_n) is monotonically increasing and bounded above. We will find a subsequence $a_{n_j} \to b$.
Because $b_1 = \inf\{a_1, a_2, ...\}$ there is a whole number n_1 with

$$b_1 \leq a_{n_1} < b_1 + \frac{1}{1}.$$

Now $b_{n_1+1} = inf\{a_{n_1+1}, a_{n_1+2}, ...\}$.
Therefore, there is an $n_2 > n_1$ such that

$$b_{n_1+1} \leq a_{n_2} < b_{n_1+1} + \frac{1}{2}.$$

Inductively define n_j such that $\forall j \in \mathbb{N}$,

$$b_{n_j+1} \leq a_{n_j} < b_{n_j+1} + \frac{1}{j}.$$

Since (b_{n_j+1}) is a subsequence of (b_n), $\lim_{j\to\infty} b_{n_j+1} = b$.
Hence, $b \leq \lim_{j\to\infty} a_{n_j} \leq b + 0$ and so $\lim_{j\to\infty} a_{n_j} = b$. □

e.g. Let (a_n) be an enumeration of $\mathbb{Q} \bigcap [0,1]$, the rational numbers between 0 and 1. Then $\forall \alpha \in \mathbb{R}$ with $0 \leq \alpha \leq 1$, there is a subsequence (a_{n_j}) (depending on α) such that $a_{n_j} \to \alpha$. This example shows just how many convergent subsequences a sequence might have.

Theorem 54 *Let $f\colon [a,b] \to \mathbb{R}$ be continuous. Then f is uniformly continuous.*

Proof. Suppose that f is **not** uniformly continuous. Then $\exists \varepsilon_0 > 0$ such that $\forall n \in \mathbb{N}$ there are numbers $x_n, y_n \in [a,b]$ such that $|x_n - y_n| < \frac{1}{n}$ but

$$|f(x_n) - f(y_n)| \geq \varepsilon_0.$$

By Thm 53, (x_n) has a convergent subsequence $x_{n_i} \to x \in [a,b]$. For the same reason, (y_{n_i}) has a convergent subsequence $y_{n_{i_j}} \to y \in [a,b]$. Then, $x_{n_{i_j}} \to x$. Because

$$|x_{n_{i_j}} - y_{n_{i_j}}| \leq \frac{1}{n_{i_j}} \leq \frac{1}{j}$$

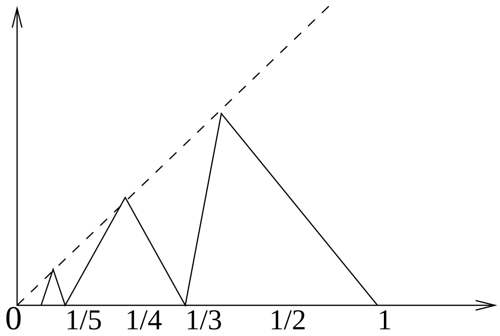
it follows that

$$|x - y| \leq 0.$$

Hence $x = y$ (note that we have implicitly invoked the continuity of the absolute value function at this point).
But f is continuous at x.
Hence

$$\lim_{j\to\infty} f(x_{n_{i_j}}) = f(x) \text{ and } \lim_{j\to\infty} f(y_{n_{i_j}}) = f(y) = f(x).$$

Therefore, since $|f(x_n) - f(y_n)| \geq \varepsilon_0$ we have $|f(x_{n_{i_j}}) - f(y_{n_{i_j}})| \geq \varepsilon_0$ and so $|f(x) - f(x)| \geq \varepsilon_0$, a contradiction.
Therefore, f is uniformly continuous on $[a,b]$. □



e.g. Let $f\colon [0,1] \to \mathbb{R}$ be the continuous function with the above graph.
That is,

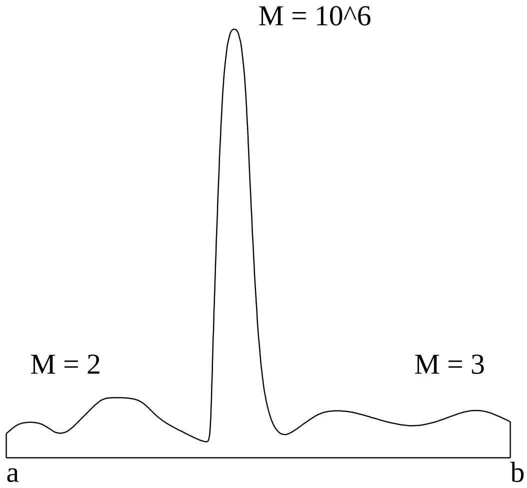
$$f(0) = 0, f(1) = 0$$

$$f(\frac{1}{2n}) = \frac{1}{2n},$$

$$f(\frac{1}{2n+1}) = 0$$

and between the points $(\frac{1}{n}, f(\frac{1}{n}), (\frac{1}{n+1}, f(\frac{1}{n+1}), f$ is linear.
Then, by Thm 23, f is uniformly continuous on [0, 1].
Exercise Find $\delta(\varepsilon)$.
However, no matter how large we choose $M \in \mathbb{R}, |f(x) - f(y)| \leq M|x - y|$ will always be false for at least one pair of points $x, y \in [0,1]$.
To see this, calculate the slope of the portion of the graph over $[\frac{1}{2n+1}, \frac{1}{2n}]$.

Later we will see that having a continuous first derivative is a sufficient condition to imply that a suitable finite $M \exists$. However, any theoretical result should be used cautiously: for a given function, M might be very large, say 10^4, because the function is very steep near one point.



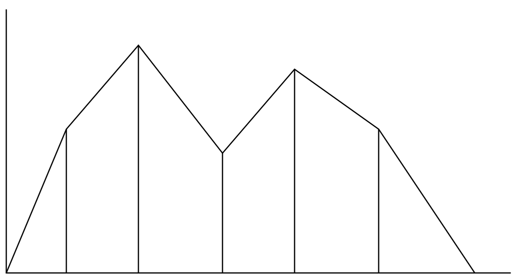
In a case such as this it would be best to split the interval into 3 parts, i.e. theoretical results based on $M = 10^6$ will be true but far from being the best possible.

Another note: the idea of a function of a single variable is very appealing in its simplicity. However, in practice, functions usually depend on parameters,

e.g.

$$f(x) = \int_a^b e^{-xs^2} \sin(Ls) ds.$$

Then $f(x) = g(a, b, L, x)$. The function is really one of four variables, the first three being parameters. For each fixed set of values of the parameters we can think of f as a function of one variable, and then all of the above theorems apply. Another example



$f(x) = h(a_1, ..., a_n, b_1, ...b_n, x)$.

16 Integration III

Theorem 55 *If $f\colon [a,b] \to \mathbb{R}$ is continuous, then f is integrable.*

Proof. Let $\varepsilon > 0$ be given. Since f is continuous on a closed and bounded interval it must be uniformly continuous. Thus there is a $\delta > 0$ such that for all $x, y \in [a,b]$ if

$$-\delta < x - y < \delta \text{ then } -\frac{\varepsilon}{b-a} < f(x) - f(y) < \frac{\varepsilon}{b-a}.$$

Let $n \in \mathbb{N}$ be such that $\frac{b-a}{n} < \delta$ and let P_n be the uniform subdivision of $[a,b]$ into n subintervals of equal width $\frac{b-a}{n}$. Then if I is any one of the subintervals and $x, y \in I$ then $|x-y| < \delta$ and so

$$f(x) < f(y) + \frac{\varepsilon}{b-a}.$$

Keep y fixed and x vary:

$$M_I(f) = max\{f(x) : x \in I\} \leq f(y) + \frac{\varepsilon}{b-a}.$$

Then let y vary:

$$M_I(f) \leq m_I(f) + \frac{\varepsilon}{b-a}$$

and so $M_I(f) - m_I(f) \leq \frac{\varepsilon}{b-a} \ \ \forall I \in P$.
Hence

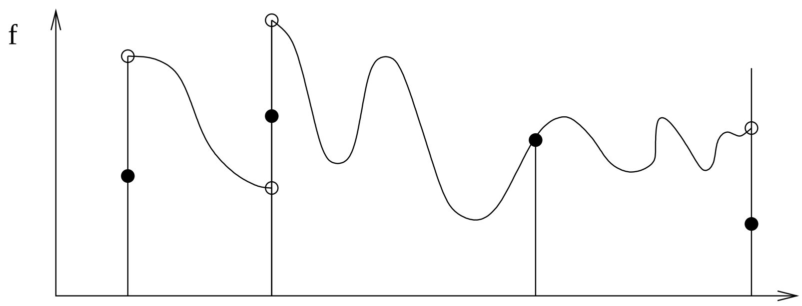
$$U(f, P_n) - L(f, P_n) = \sum_I (M_I(f) - m_I(f))\Delta I \leq (\frac{\varepsilon}{b-a}) \cdot (b-a) = \varepsilon.$$

Thus, f is integrable. □

By Thm 55, functions like $x \to x^2 + 2$, $x \to \sin(x^2 + x)$, $x \to \frac{1}{x}$ are integrable on closed bounded intervals upon which they are continuous. We aim to show that "piecewise continuous" bounded functions are integrable. A suitable definition of piecewise continuous functions is as follows: A function $f\colon [a,b] \to \mathbb{R}$ is piecewise continuous if $\exists$ points

$$a = a_0 < a_1 < a_2 < ... < a_n = b$$

in $[a, b]$ such that $\forall i$ with $1 \leq i \leq n$
i. f is continuous on (a_{i-1}, a_i)
ii. $\lim\limits_{x \to a_{i-1}} f(x)$ and $\lim\limits_{x \to a_i -} f(x)$ both exist as finite real numbers.



To show such functions are integrable we first show how we may 'join' integrable functions together and 'take them apart'.

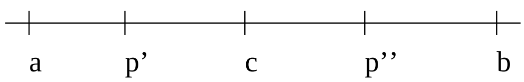
Theorem 56 *Let f be integrable on $[a, c]$ and $[c, b]$ where $a < c < b$. Then f is integrable on $[a, b]$ and $\int_a^b f = \int_a^c f + \int_c^b f$.*

Proof. Let $\varepsilon > 0$ be given. There is a partition P' of [a, c] such that

$$U(f, P') - L(f, P') < \frac{\varepsilon}{2}.$$

There is a partition P'' of $[c, b]$ such that

$$U(f, P'') - L(f, P'') < \frac{\varepsilon}{2}$$



If $P' = (x_0, ..., x_n)$ and $P'' = (y_0, ..., y_m)$ let a partition P of $[a, b]$ be defined by

$$P = (x_0, x..., x_n, y_1, ...y_m) = P \bigcup P'' \text{ (by definition).}$$

Then $U(f, P) = U(f, P') + U(f, P'')$
and $L(f, P) = L(f, P') + L(f, P'')$ and so

$$U(f, P) - L(f, P) = U(f, P') - L(f, P') + U(f, P'') - L(f, P'') < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Therefore, f is integrable on $[a, b]$.

Since $U(f, P) = U(f, P') + U(f, P'')$
and $U(f, P') < \frac{\varepsilon}{2} + \int_a^c f$ and $U(f, P'') < \frac{\varepsilon}{2} + \int_c^b f$
it follows that $\int_a^b f \leq U(f, P) < \varepsilon + \int_a^c f + \int_c^b f$.

Since this is true $\forall \varepsilon > 0$ we obtain $\int_a^b f \leq \int_a^c f + \int_c^b f$. The opposite inequality is obtained in a similar manner starting with $L(f, P) = L(f, P') + L(f, P'')$ (try this as an exercise).

Therefore, $\int_a^b = \int_a^c f + \int_c^b f$. □

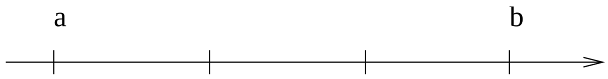
The above very useful result is probably one you have used many times but never seen proved. The same applies to the next theorem.

Theorem 57 *Let f be integrable on $[a, b]$ and let $a \leq \alpha \leq \beta \leq b$. Then f is integrable on $[\alpha, \beta]$.*

Proof. Let $\varepsilon > 0$ be given and let P' be a partition of $[a, b]$ satisfying

$$U(f, P') - L(f, P') < \varepsilon.$$

Let P'' be the partition (a, α, β, b) and let $P = P' \bigcup P''$. Then, since P refines $P', U(f, P) - L(f, P) < \varepsilon$.



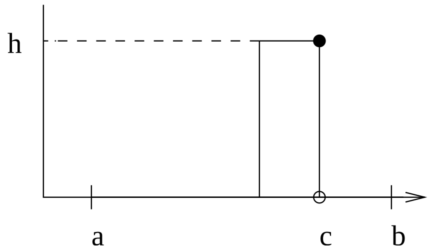
Let $P = (x_0, ..., x_n)$ and suppose $\alpha = x_l, \beta = x_m$.

Then $\varepsilon > I(f, P) - L(f, P) = \sum_{i=1}^{n} (M_i(f) - m_i(f))\Delta x_i$

$$\geq \sum_{i=l+1}^{m} (M_i(f) - m_i(f))\Delta x_i$$

since each term is positive.
Let a partition Q of $[\alpha, \beta]$ be defined by $Q = (x_1, ..., x_m)$.
Then $U(f, Q) - L(f, Q) < \varepsilon$ is just a restatement of the given inequality. It shows f is integrable on $[\alpha, \beta]$. □

e.g. The following example shows that we can change the values of a function at a finite number of points and not alter the value of its integral.



By Thm 56 it is sufficient to consider the example of a function $f : [a, c] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 0 & \text{for } a \le x < c \\ h & \text{for } x = c \end{cases}$$

We will prove the integral of this function is zero.
Let P_n be the uniform partition of $[a, c]$. Then

$$U(f, P_n) = \sum_{i=1}^{n} M_i(f)\frac{(c-a)}{n} = 0 + h\frac{(c-a)}{n}$$

and

$$L(f, P_n) = 0 \;\; \forall n \in \mathbb{N}.$$

Since $U(f, P_n) - L(f, P_n) \le \frac{h(c-a)}{n}$, f is integrable.

Thus

$$L(f, P_n) = 0 \le \int_a^c f \le \frac{h(c-a)}{n} = U(f, P_n) \;\; \forall n \in \mathbb{N}.$$

Letting $n \to \infty$ we obtain $\int_a^c f = 0 = \int_a^c h_0$. Similarly, $\int_c^b f = 0$ and hence, by Thm 52, $\int_a^b f = 0$. If $f = 0$ everywhere on $[a, b]$ except at $c_1, ..., c_n$ where it has values $h_1, ..., h_n$ respectively let functions f_i be defined on $[a, b]$ by

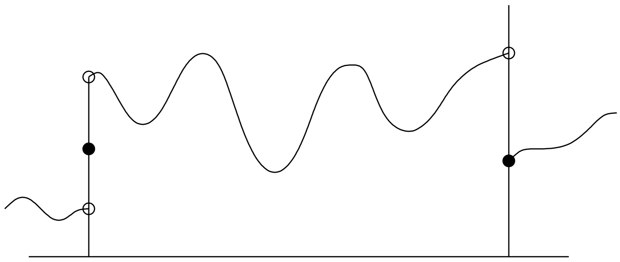
$$f_i(x) = 0 \;\; \forall x \in [a, b] \setminus \{c_i\}$$

$$f_i(c_i) = h_i.$$

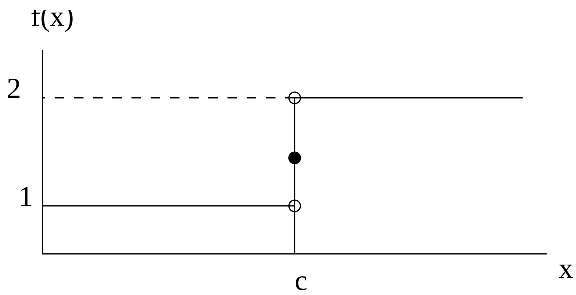
Then $f(x) = \sum_{i=1}^{n} f_i(x) \;\; \forall x \in [a, b]$ and so,

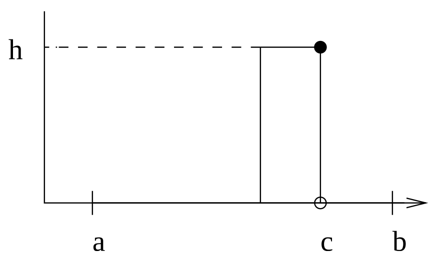
$$\int_a^b f = \sum_{i=1}^{n} \int_a^b f_i = \sum_{i=1}^{n} 0 = 0.$$

Finally, let $f\colon [a,b] \to \mathbb{R}$ be integrable and equal to g except at a finite number of points. Then $f-g$ is zero except at a finite number of points (and so g is integrable) and $\int(f-g)=0$. But then $\int f = \int g$.
Now let $f\colon [a,b] \to \mathbb{R}$ be piecewise continuous and let $a=a_0<a_1<\ldots<a_n=b$ be the partition of $[a,b]$ described near the beginning of the lecture.



Define a continuous function $g_i : [a_{i-1},a_i] \to \mathbb{R}$ as follows:
If $a_{i-1}<x<a_i$ then $g_i(x)=f(x)$.
If $x=a_{i-1}$ then $g_i(x)=\lim\limits_{x\to a_{i-1}+} f(x)$.
If $x=a_i$ then $g_i(x)=\lim\limits_{x\to a_{i-1}-} f(x)$.
Then, by the above discussion, since g_i is integrable for $1 \le i \le n, f$ is integrable on $[a_{i-1},a_i]$ for $1 \le i \le n$ and so, f is integrable on $[a,b]$. To complete this extended example one should look at the way the integral changes in value as the endpoint b goes through a discontinuity of f:





17 Integration IV

Theorem 58 *Let $f\colon [a,b] \to \mathbb{R}$ be integrable and let $F\colon [a,b] \to \mathbb{R}$ be defined by*

$$F(y) = \int_a^y f \quad \text{for } a \le y \le b.$$

Then F is continuous on $[a,b]$.

Proof. Since f is bounded there is a constant M such that

$$|f(x)| \le M \;\; \forall x \in [a,b].$$

Then if y and $y+h \in [a,b]$ and $h > 0$

$$|F(y+h) - F(y)| = |\int_a^{y+h} f - \int_a^y f| = |\int_y^{y+h} f|$$

by Thm 56.

$$\le \int_y^{y+h} |f|.$$

by Thm 52

$$\le \int_y^{y+h} M$$

$$= M(y+h-y)$$

$$= Mh \le M|h|.$$

Therefore, $|F(y+h) - F(y)| \le M|h|$. The same is true when $h < 0$ and the continuity of F is a direct consequence of this inequality. □

Note that it also implies that the Newtonian quotients for F at various points $y \in [a,b]$ are all bounded by M. This is true even where F is not differentiable. This can only occur where f is discontinuous

Theorem 59 *Let $f\colon [a,b] \to \mathbb{R}$ be* **continuous** *and let F be defined as above. Then F is differentiable and $\forall y \in (a,b)$,*

$$F'(y) = f(y)$$

i.e. F is an antiderivative for f.

Proof. Let $y \in (a,b)$. Given $\varepsilon > 0, \exists \delta > 0$ such that if $|x-y| < \delta$, then $x \in (a,b)$ and $|f(x) - f(y)| < \varepsilon$. Then if $0 < |x-y| < \delta$,

$$\begin{aligned}
|\frac{F(x)-F(y)}{x-y} - f(y)| &= \frac{|F(x)-F(y)-f(y)(x-y)|}{x-y} \\
&= \frac{|\int_a^x f - \int_a^y f - f(y)\int_y^x 1|}{|x-y|} \\
&= \frac{|\int_y^x f + \int_y^x (-f(y))dx|}{|x-y|} \\
&= \frac{|\int_y^x f(x) - f(y)dx|}{|x-y|} \\
&\leq \frac{\int_y^x |f(x) - f(y)|dx}{|x-y|} \\
&\leq \varepsilon \frac{|x-y|}{|x-y|} = \varepsilon
\end{aligned}$$

Therefore $F'(y) = f(y)$. □

One can show similarly that F is differentiable on the right at a and on the left at b with $F'_+(a) = f(a)$ and $F'_-(b) = f(b)$.

Theorem 60 Fundamental Theorem of Calculus
Let $f\colon [a,b] \to \mathbb{R}$ be continuous and let $G\colon [a,b] \to \mathbb{R}$ satisfy $G'(y) = f(y)$ on (a,b). Then

$$\int_a^b f = G(b) - G(a).$$

Proof. With the notation used in Thm 58, since $F' = G'$ on (a,b) and both are continuous (F because of Thm 57 and G because it is differentiable), by the Mean Value Thm they must differ at most by a constant C, say, i.e.

$$G(x) = F(x) + C \text{ on } [a,b].$$

Since $F(a) = \int_a^a f = 0$, it follows that $G(a) = C$ and thus $F(x) = G(x) - G(a)$. Finally let $x = b$:

$$\int_a^b f = F(b) = G(b) - G(a).$$

It follows from Thm 59 that any continuous function f has an antiderivative F. However functions which are derivatives of other functions are not necessarily themselves continuous. However they might manage to be integrable even if not continuous; in this case a result similar to that of Thm 60 continues to be true as we will show below. $\square$

e.g.

$$f(x) = \begin{cases} x^2\sin(\frac{1}{x}) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Then f is differentiable if $x \neq 0$ and $f'(x) = 2x\sin(\frac{1}{x}) - \cos(\frac{1}{x})$.

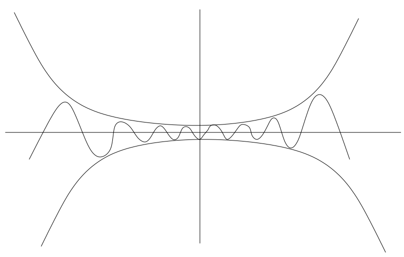
If $x = 0$: $f'(0) = \lim_{h\to 0} \dfrac{f(h) - f(0)}{h}$

$$= \lim_{h\to 0} \frac{h^2\sin(\frac{1}{h})}{h}$$

$$= \lim_{h\to 0} h\sin(\frac{1}{h}) = 0 \quad \text{since} |\sin(\frac{1}{h})| \leq 1.$$

But $\lim_{h\to 0} f'(h) \nexists$. Thus f' is not continuous at 0. Note however that f itself must be continuous since it is differentiable.

$$y = x^2\sin(\frac{1}{x})$$



Theorem 61 *Let $g\colon [a,b] \to \mathbb{R}$ be differentiable and suppose also that g' is integrable on$[a,b]$. Then*

$$\int_a^b g' = g(b) - g(a).$$

Proof. Let P be any partition of $[a,b], P=(x_0,...,x_n)$.
Then

$$g(b)-g(a)=\sum_{j=0}^{n}(g(x_j)-g(x_{j-1}))$$

$$=\sum_{j=0}^{n}g'(\xi_j)(x_j-x_{j-1})$$

applying the Mean Value Theorem to g on $[x_{j-1},x_j]$, where ξ_j is some point in (x_{j-1},x_j). But if

$$m_j=inf\{g'(x):x\in I_j\}$$

and

$$M_j=sup\{g'(x):x\in I_j\}$$

then

$$m_j\le g'(\xi_j)\le M_j$$

and so

$$L(g',P)=\sum m_j\Delta x_j\le g(b)-g(a)\le\sum M_j\Delta x_j=U(g',P).$$

Since g' is integrable there is a P with $U(g',P)-\varepsilon\le\in g'\le L(g',P)+\varepsilon$. Hence

$$\int_a^b g'=g(b)-g(a).$$

□

Note that it is not easy to construct a differentiable function g with g' not continuous. Thus Thm 61 is a proper but not very useful extension of Thm 60. It is that theorem which gives rise to one of the most frequently used methods for evaluating integrals.

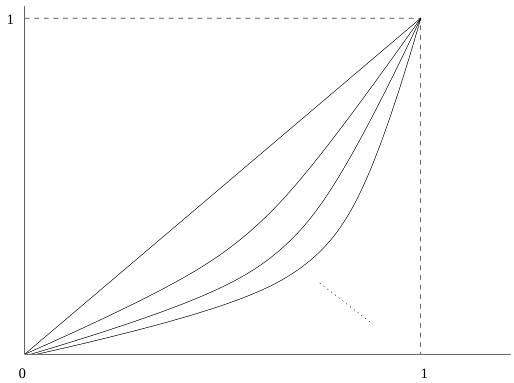
17.1 Sequences of Functions

These have occurred already. For example if:

$$f_n(x)=1+x^2+x^3+...+x^n \text{ then } f_n\to f$$

where $f(x) = \frac{1}{1-x}$ on (-1, 1).

Consider another example $f_n(x) = x^n, n \in \mathbb{N}$ on [0, 1].



Let

$$f(x) = \begin{cases} 0 & \text{if } x < 1 \\ 1 & \text{if } x = 1 \end{cases}$$

Then for each x in [0, 1], $\lim_{n\to\infty} f_n(x) = f(x)$. We say $f_n \to f$ pointwise on [0, 1]. Consider the integrals

$$\int_0^1 f_n = [\frac{x^{n+1}}{n+1}]_0^1 = \frac{1}{n+1}$$

$$\int_0^1 f = 0$$

and $\lim_{n\to\infty} \int_0^1 f_n = 0 = \int_0^1 \lim_{n\to\infty} f_n$, where the last limit is the pointwise limit. We would like to be able to do this in general but cannot because the pointwise limit might not even be integrable (in this example it is not continuous but is integrable). In order that we might be able to interchange the limit and the integral we need a stronger form of convergence. There are various 'monotone convergence' results (the above is one example) but we will develop the more useful **uniform convergence** notion:

Definition A sequence of functions (f_n) on a set S is said to **converge uniformly** on S to a function f is $\forall \varepsilon > 0 \; \exists N_\varepsilon \in \mathbb{N}$ such that $\forall n \geq N_\varepsilon$,

$$|f(x) - f_n(x)| < \varepsilon \;\; \forall x \in S.$$

Note that this means N_ε is independent of x in S.

Definition A sequence of functions (f_n) is said to **converge pointwise** on S to a function f if $\forall\varepsilon > 0$, for each $x \in S$, $\exists N(\varepsilon, x) \in \mathbb{N}$ such that

$$|f(x) - f_n(x)| < \varepsilon \;\; \forall n \geq N(\varepsilon, x).$$

Exercise Let a function $f : [0, 4] \to \mathbb{R}$ be defined by $f(0) = 1. f(x) = x^2$ if $0 < x < 1, f(1) = 0, f(x) = 2 - x$ if $1 < x < 3, f(3) = 0, f(x) = 2x - 4$ if $3 < x \leq 4$.
a. Plot the graph of f.
b. Evaluate $F(x) := \int_0^x f$ as an explicit function of x (defined piecewise).
c. Plot the graph of F.
d. Calculate F' and compare its graph with that of f.
e. Calculate $\int_0^x F'$ as a function of F.
f. Where does this process stop and why?

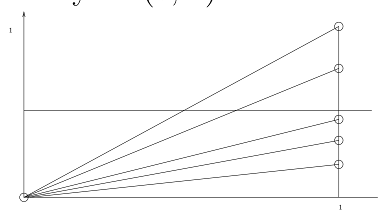
18 Uniform Convergence of Function Sequences

e.g.
1. Let $f_n(x) = \frac{x}{n}$ for $x \in (0, 1) \quad n = 1, 2, 3, \ldots$.
Pointwise convergence: $\lim\limits_{n\to\infty} f_n(x) = \lim\limits_{n\to\infty} \dfrac{x}{n} = x \lim\limits_{n\to\infty} \dfrac{1}{n} = 0$.
Therefore $f_n \to h_0$ pointwise on (0, 1).

Uniform convergence: Given $\varepsilon > 0$, let N_ε satisfy $N_\varepsilon > \frac{1}{\varepsilon}$.
Then if $n \geq N_\varepsilon$, $|f_n(x) - h_0(x)| = |f_n(x) - 0|$

$$= \frac{|x|}{n} < x \cdot \varepsilon < \varepsilon \;\; \forall x \in (0, 1).$$

Therefore, $f_n \to h_0$ uniformly on (0, 1).



2. Let $f_n(x) = \frac{1}{nx}$ on $(0,1)$ $\quad n = 1, 2, 3, \ldots$.
Pointwise convergence: $\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \dfrac{1}{nx} = \dfrac{1}{x} \lim_{n\to\infty} \dfrac{1}{n} = 0$.

Uniform convergence: Take $\varepsilon = 1$.
Does $\exists\, N_1$ such that $\forall n \geq N_1$ and $\forall x \in (0,1), |f_n(x) - 0| < 1$?
This is true if and only if $\frac{1}{nx} < 1 \ \ \forall x \in (0,1)$ and $\forall n \geq N_1$.
This is impossible - let $x = \frac{1}{N_1+1}$, then $0 < x < 1$ but

$$\frac{1}{nx} = \frac{N_1 + 1}{N_1} > 1 = \varepsilon.$$

Hence f_n does not converge uniformly to h_0 nor to any other function since it converges pointwise to h_0.

3. If each f_n is continuous (even C^∞) and $f_n \to f$ pointwise on S the limit function f will not in general be continuous (it might not even be integrable). Let $f_n(x) = x^n$ on [0, 1].
Then

$$\lim_{n\to\infty} f_n(x) = \begin{cases} 0 & \text{if } x < 1 \\ 1 & \text{if } x = 1 \end{cases}$$

and so the pointwise limit is the discontinuous function

$$f(x) = \begin{cases} 0 & \text{if } x < 1 \\ 1 & \text{if } x = 1 \end{cases}$$

This can never happen if $f_n \to f$ uniformly.

Theorem 62 *Let $f_n \colon S \to \mathbb{R}$ be a sequence of continuous functions and suppose that $f_n \to f$* **uniformly** *on S. Then f is continuous on S.*

Proof. Given $\varepsilon > 0$ and $\xi \in S$ we need to show f is continuous at ξ, i.e. $\exists \delta > 0$ such that if $x \in S$ and $|x - \xi| < \delta$ then $|f(x) - f(\xi)| < \varepsilon$. Since $f_n \to f$ uniformly there is an $N_\varepsilon \in \mathbb{N}$ for which

$$n \geq N_\varepsilon \implies |f_n(x) - f(x)| < \frac{\varepsilon}{3} \ \ \forall x \in S.$$

Since f_n is continuous at ξ there is a $\delta_n > 0$ such that if $|x - \xi| < \delta_n$ then $|f_n(x) - f_n(\xi)| < \frac{\varepsilon}{3}$. Let $\delta_\varepsilon = \delta_{N_\varepsilon}$. If $|x - \xi| < \delta_\varepsilon$ then

$$|f(x) - f(\xi)| \leq |f(x) - f_{N_\varepsilon}(x)| + |f_{N_\varepsilon}(x) - f_{N_\varepsilon}(\xi)| + |f_{N_\varepsilon}(\xi) - f(\xi)|$$

$< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$ □

Theorem 63 *Let $f_n \colon [a,b] \to \mathbb{R}$ for $n \in \mathbb{N}$ be integrable functions and let $f_n \to f$ uniformly on $[a,b]$. Then f is integrable on $[a,b]$ and*

$$\lim_{n\to\infty} \int_a^b f_n(x)\,dx = \int_a^b \lim_{n\to\infty} f_n(x)\,dx = \int_a^b f.$$

Proof. Let $\varepsilon > 0$ be given. There is an $n \in \mathbb{N}$ such that

$$|f_n(x) - f(x)| < \frac{\varepsilon}{2(b-a)} \quad \forall x \in [a,b].$$

Thus $f_n(x) - \frac{\varepsilon}{2(b-a)} < f(x) < f_n(x) + \frac{\varepsilon}{2(b-a)}$. Note that if g, h, k are three bounded functions satisfying $g \leq h \leq k$ and if P is any partition then

$$L(g,P) \leq L(h,P) \leq U(h,P) \leq U(k,P).$$

Since f_n is integrable there is a partition P_n of $[a,b]$ such that

$$U(f_n, P_n) - L(f_n, P_n) < \frac{\varepsilon}{2}.$$

It follows then that

$$L(f_n, P_n) - \frac{\varepsilon}{2} \leq L(f, P_n) \leq U(f, P_n) \leq U(f_n, P_n) + \frac{\varepsilon}{2}.$$

Hence $U(f, P_n) - L(f, P_n) \leq U(f_n, P_n) - L(f_n, P_n) + \varepsilon < 2\varepsilon$ and thus f is integrable. □

Finally we will show that the limit of the integrals is the integral of the limit.

Given $\varepsilon > 0 \exists N_\varepsilon$ such that $\forall n \geq N_\varepsilon$,

$$|f_n(x) - f(x)| < \frac{\varepsilon}{b-a} \forall x \in [a,b].$$

Then for $n \geq N_\varepsilon$ we have

$$|\int_a^b f_n - \int_a^b f| = |\int_a^b f_n - f| \leq \int_a^b |f_n - f| \leq \int_a^b \frac{\varepsilon}{b-a} = \varepsilon.$$

Thus $\lim_{n\to\infty}\int_a^b f_n = \int_a^b f$ and the proof is complete.

e.g. Let $f_n(x) = \frac{2nx}{1+n^2x^4}$, $n = 1, 2, 3, ...$ on [0, 1].

$$\lim_{n\to\infty} f_n(x) = 0 = h_0(x) \ \ \forall x \in [0,1] \text{ and } \int_0^1 h_0 = 0.$$

But

$$\begin{aligned}\lim_{n\to\infty}\int_0^1 f_n &= \lim_{n\to\infty}\int_0^1 \frac{2nx\,dx}{1+n^2x^4}\\ &= \lim_{n\to\infty}[\arctan(nx^2)]_0^1\\ &= \lim_{n\to\infty}[\arctan(n)] = \frac{\pi}{2} \neq 0.\end{aligned}$$

Thus the convergence of $\frac{2nx}{1+n^2x^4}$ to 0 is not uniform.

Theorem 64 *Let $f_n\colon [a,b] \to \mathbb{R}$ be a sequence of functions each having a continuous first derivative ($f_n \in C^{(1)}[a,b]$).*
Let the sequence of derivatives $f_n' \to g$ **uniformly** *on $[a,b]$.*
Let the original sequence $f_n \to f$ **pointwise** *on $[a,b]$.*
Then f is differentiable and $f'(x) = f(x)$ for $a < x < b$, i.e. $f_n' \to f'$.

Proof. Because $f_n' \to g$ uniformly on $[a,b]$ the same is true on $[a,x]$ for $a \le x \le b$. Therefore, by Thm 63,

$$\lim_{n\to\infty}\int_a^x f_n' = \int_a^x g.$$

Note that each f_n' being continuous (and thus integrable) so is g continuous (and so integrable). Because f_n' is integrable,

$$\int_a^x f_n' = f_n(x) - f_n(a).$$

Thus

$$\lim_{n\to\infty} f_n(x) - f_n(a) = f(x) - f(a) = \int_a^x g.$$

Since g is continuous, Thm 58 implies $\int_a^x g$ is differentiable. Thus the LHS is differentiable (in fact continuously differentiable) and

$$f'(x) - 0 = g(x).$$

□

In applications the sequences of functions (f_n) will almost always arise as the partial sums of a series of functions; for example

$$\exp x = 1 + x + \frac{x^2}{2!} + ... + \frac{x^n}{n!} + ...$$

$$= \sum_{n=0}^{\infty} f_n(x) = \lim_{n\to\infty} f_n(x)$$

where $g_n(x) = \frac{x^n}{n!}$ and $f_n(x) = \sum_{j=0}^{n} g_j(x)$.

We say the series $\sum_1^\infty g_n$ **converges uniformly to** f **on** S if (f_n), the sequence of partial sums, converges uniformly to f on S. Because each g_n integrable, continuous, differentiable, etc implies each f_n integrable, continuous, differentiable, etc respectively, all of the above theorems have very easily proved extensions to series of functions, i.e.

Theorem 65 *If each* g_n *is continuous and* $\sum_1^\infty g_n = f$ *uniformly on* S *then* f *is continuous on* S.

Theorem 66 *If each* g_n *is integrable and* $\sum_1^\infty g_n = f$ *uniformly on* $[a, b]$, *then* f *is integrable and*

$$\sum_1^\infty \int_a^b g_n = \int_a^b f.$$

Theorem 67 *If each* $g_n \in C^{(1)}[a, b]$ *and* $\sum g_n' \to g$ *uniformly and* $\sum g_n = f$ *pointwise then* $f' = g$ *on* (a, b).

For example, consider 65: If $f_n = g_1 + ... + g_n$ and the g_n are continuous, then so if f_n. Since $\sum_1^\infty g_n = f$ uniformly, by definition $f_n \to f$ uniformly. Thus, by Thm 64, f is continuous.

Exercises

1. Let $h_n : [0,1] \to \mathbb{R}$ be defined by

$$h_n(x) = \begin{cases} 1 - nx, \ 0 \le x \le \frac{1}{n} \\ 0, \ \frac{1}{n} < x \le 1 \end{cases}$$

Consider $\lim h_n$.

2. If $f_n(x) = (\cos \pi x)^{2n}$ on $\mathbb{R}$, consider $\lim f_n$.

3. If $f_n(x) = \frac{2}{\pi} \arctan(nx)$ evaluate $\lim f_n$.

4. Does the sequence (xe^{-nx}) converge uniformly for $x \ge 0$?

5. Does the sequence (x^2e^{-nx}) converge uniformly for $x \ge 0$?

6. Define

$$g_n(x) = \begin{cases} nx, \ 0 \le x \le \frac{1}{n} \\ \frac{n(1-x)}{n-1}, \ \frac{1}{n} < x \le 1 \end{cases}$$

Show that $\lim g_n \ \exists$ pointwise on $[0,\, 1]$. Is it uniform on $[c,\, 1]$ for $0 < c < 1$?

19 Exponential and Natural Logarithm Functions

Let $x \in \mathbb{R}$ and let $\exp x = \displaystyle\sum_0^\infty \frac{x^n}{n!}$. Then, for fixed x, the series converges pointwise on $\mathbb{R}$. The convergence is absolute.

Theorem 68 *If $A \ge 0$ then the series $\exp x$ converges uniformly on $[-A, A]$.*

Proof. By the above remark, $\displaystyle\sum_0^\infty \frac{A^n}{n!} < \infty$. Let $\varepsilon > 0$ be given. Then, $\exists N_\varepsilon \in \mathbb{N}$ such that $\forall m \ge N_\varepsilon, \displaystyle\sum_{m+1}^\infty \frac{A^n}{n!} < \varepsilon$. Let $x \in [-A, A]$. Then $\forall i, |x|^i \le A^i$ and therefore $|\displaystyle\sum_{m+1}^l \frac{x^i}{i!}| \le \sum_{m+1}^l \frac{|x|^i}{i!} \le \sum_{m+1}^\infty \frac{A^n}{n!} < \varepsilon$. Thus the series converges uniformly because N_ε is independent of x in $[-A, A]$. □

Note

1. The series does not converge uniformly on $\mathbb{R}$.
2. Because A is arbitrary the uniform convergence on $[-A, A]$ is sufficient to show that $\exp x$ is continuous on $\mathbb{R}$ (Exercise).

Theorem 69 *If* $x, y \in \mathbb{R}, \exp(x+y) = \exp x \cdot \exp y$.

Proof. The series for $\exp x$ and $\exp y$, by the above remarks, converge absolutely. Therefore,

$$(\exp x)(\exp y) = (\sum_0^\infty \frac{x^n}{n!})(\sum_0^\infty \frac{y^n}{n!}) = \sum_0^\infty c_n$$

where

$$c_n = \frac{x^n}{n!}1 + \frac{x^{n-1}}{(n-1)!} \cdot \frac{y}{1!} + \frac{x^{n-2}}{(n-2)!} \cdot \frac{y^2}{2!} + \frac{x^{n-r}}{(n-r)!} \cdot \frac{y^r}{r!} + ... + \frac{1 \cdot y^n}{n!}$$

$$= \frac{1}{n!}[x^n + \frac{n!x^{n-1}y}{(n-1)!1!} + ... + \frac{n!x^{n-r}y^r}{(n-r)!r!} + ... + y^n] = \frac{1}{n!}(x+y)^n$$

by the Binomial Theorem.

Thus $\sum_0^\infty c_n = \exp(x+y)$. □

19.1 Derivative of $\exp x$

We have shown that exp is continuous on $\mathbb{R}$ by invoking the uniform convergence of the exp series. Unfortunately, as was pointed out earlier, we **cannot** say $\exp x$ is differentiable because it is the uniform limit of differentiable functions (The Stone-Wierstrass Thm, not in this course, says that any continuous function on any interval $[a, b]$, is the uniform limit of a sequence of polynomials; in particular non-differentiable functions may be limits of differentiable polynomials). There are conditions (Thm 63) under which when a sequence of functions $f_n \to f$ then the sequence of derivatives $f_n' \to f'$ the derivative of the limit function. The theorem implies that the limit function f is differentiable. However we may prove that $\exp x$ is differentiable from first principles:

Theorem 70 $\forall x \in \mathbb{R}, \exp x$ *is differentiable and* $\exp'(x) = \exp x$.

Proof.

$$|\frac{\exp(x+h) - \exp x}{h} - \exp x| = \frac{1}{|h|}|\exp x \cdot \exp h - \exp x - \exp xh|$$

$$= \frac{|\exp x|}{|h|}|\exp x - 1 - h|$$

$$\exp h - 1 - h = \frac{h^2}{2!} + \frac{h^3}{3!} + \frac{h^4}{4!} + ...$$

Thus,

$$|\frac{\exp h - 1 - h}{h}| \leq |h|\{\frac{1}{2!} + \frac{|h|}{3!} + \frac{|h|^2}{4!} + ...\}$$

$$< |h|\{\frac{1}{2!} + \frac{1}{3!} + ...\} \text{ if} |h| < 1$$

$$< \exp 1 \cdot |h|.$$

Therefore

$$\lim_{h \to 0} |\frac{\exp(x+h) - \exp x}{h} - \exp x| \leq \lim_{h \to 0} \exp(1)|h|$$

$$\Rightarrow \exp'(x) = \exp x.$$

We shall now apply Thm 64 to the exponential function.
Let $f_n(x) = \frac{x^n}{n!}$ $n = 0, 1, 2, ...$ Then $f_n'(x) = \frac{x^{n-1}}{(n-1)!}, n = 1, 2, ...$ and $\sum_{n=1}^{\infty} f_n'(x) = \exp x$ uniformly on $[-A, A]$ if $A > 0$.
Clearly $f_n \in c^{(1)}[-A, A]$ for each n.
$\therefore \exp x = \sum_{n=1}^{\infty} f_n'(x) = \exp'(x)$ and we have proved the result once more. □

19.2 Properties of exp

From the series, if $x \geq 0$ then $\exp x \geq 1 > 0$. We know, from an earlier result, that $\exp(x + y) = \exp x \cdot \exp y$.

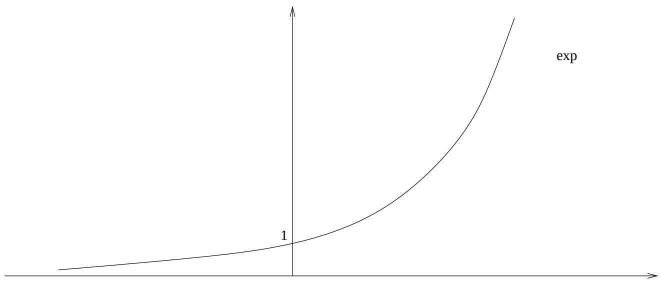
$$\therefore\ 1 = \exp 0 = \exp x \cdot \exp(-x) \Rightarrow \exp(-x) = \frac{1}{\exp x} > 0 \text{ if} x \geq 0.$$

i. Therefore, $\exp x > 0 \ \ \forall x \in \mathbb{R}$.
ii. Also,

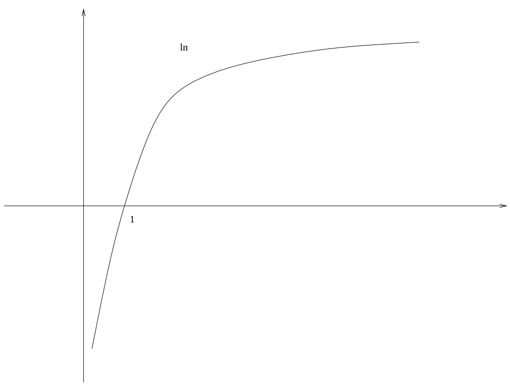
$$\exp'(x) = \exp x \Rightarrow \frac{d}{dx} \exp x > 0 \ \ \forall x \in \mathbb{R}.$$

By the Mean Value Theorem, if $a < b$ then $\exp b - \exp a = (b-a)\exp'(\xi)$ where $a < \xi < b$. Thus $\exp b > \exp a$ and exp is
iii. increasing on $\mathbb{R}$.
iv. $\lim_{x\to\infty} \exp x \geq \lim_{x\to\infty} (1 + \frac{x^n}{n!}) = \infty$.
v. $\lim_{x\to\infty} \exp(-x) = \lim_{x\to\infty} = \lim_{x\to\infty} \frac{1}{\exp x} = 0$. Thus $\lim_{x\to-\infty} = 0$.

From these properties we can draw the graph of the function:



They also indicate that $\exp : \mathbb{R} \to (0,\infty)$ is a one-to-one and onto function and therefore there exists an inverse function $\ln : (0,\infty) \to \mathbb{R}$ which is also one-to-one and satisfies $x = \exp(\ln x) \;\; \forall x \in (0,\infty), x = \ln(\exp x) \;\; \forall x \in \mathbb{R}$. The graph of ln can be drawn by reflecting the graph of exp about the line $y = x$.
$0 = \ln(\exp 0) = \ln 1$.



We may calculate the successive derivatives of $\ln x$ using the formula for the derivative of the inverse function proved in First year Calculus: namely

$$(f^-1)'(x) = \frac{1}{f'(f^-1(x))}.$$

Then

$$\ln'(x) = \frac{1}{\exp'(\ln x)} = \frac{1}{\exp(\ln x)} = \frac{1}{x}$$

when $x \in$ domain $\ln = (0, \infty)$. Therefore

$$\ln^{(n)}(x) = \frac{(-1)^{n-1}(n-1)!}{x^n}, \quad n = 1, 2, 3, ...$$

(you should easily verify this formula for $n = 2$ and 1 and prove the general case by induction on n).

This shows that all the derivatives of ln exist and are continuous when $x \in (0, \infty)$.
Thus we can apply Taylor's Thm in the case $a = 1$ and $h = x \in (-1, \infty)$:

$$f(1 + x) = f(1) + \frac{f'(1)}{1!}x + \frac{f''(1)}{2!}x^2 + ... + \frac{f^{(n-1)}(1)}{(n-1)!}x^{n-1} + R_n$$

where $R_n = \frac{f^{(n)}(\xi)}{n!}x^n$ for some $\xi \in (1 + 1 + x)$ if $x \geq 0$ or in $(1 + x, 1)$ if $x \leq 0$. Therefore

$$\ln(1 + x) = \ln(1) + x + \frac{(-1)1!}{2!}x^2 + ... + \frac{(-1)^{n-2}(n-2)!}{(n-1)!x^{n-1}} + R_n$$

$$= x - \frac{x^2}{2} + \frac{x^3}{3} - ... + \frac{(-1)^n}{n-1}x^{n-1} + R_n.$$

Also,

$$|R_n| = |\frac{(-1)^{n-1}(n-1)!x^n}{n!\xi^n}| = \frac{1}{n}|\frac{x}{\xi}|^n.$$

Note that ξ depends on x and on n, i.e. $\xi = \xi(x, n)$.
If $0 \leq x \leq 1$ then $x \leq 1 < \xi < 1 + x \leq 2$ and so $|\frac{x}{\xi}| \leq 1$, therefore $\lim_{n\to\infty} R_n = 0$ and we can say $\ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - ...$ pointwise when

$x \in [0,1]$.
However, one cannot prove pointwise convergence for other values of x using this method. We will use a general theorem, which follows, to prove uniform converge once some intervals.

Theorem 71 Wierstrass M-test
Let $f_n : S \to \mathbb{R}$ be a sequence of functions and suppose there exists a sequence of positive real numbers $\{M_n\}$ satisfying $M_n \geq 0, |f_n(x)| \leq M_n \; \forall x \in S$ and $\sum_{n=1}^{\infty} M_n < \infty$. Then $\sum_{1}^{\infty} f_n(x)$ is uniformly convergent on S.

Proof. Because $\sum M_n$ is convergent, given $\varepsilon > 0, \exists N_\varepsilon \in \mathbb{N}$ such that $\forall m \geq l > N_\varepsilon, |\sum_{n=l}^{m} M_n| < \varepsilon$, because convergence $\Rightarrow$ Cauchy convergence. Thus

$$|\sum_{n=l}^{m} f_n(x)| \leq \sum_{n=l}^{m} |f_n(x)| \leq \sum_{n=l}^{m} M_n = |\sum_{n=l}^{m} M_n| < \varepsilon \;\; \forall x \in S.$$

Thus, by the Cauchy test for uniform convergence, $\sum_{1}^{\infty} f_n(x)$ converges uniformly on S. □

Consider the series $f(x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots$. We have seen that the series converges pointwise to $\ln(1+x)$ for each $x \in [0,1]$. We will show that it converges uniformly on $[-A, A]$ for any number A satisfying $0 \leq A < 1$.

Let $M_n = A^n$.
Then $\sum_{1}^{\infty} M_n = \sum_{1}^{\infty} A^n = \dfrac{A}{1-A} < \infty$ (sum to infinity of a geometric progression) and $\forall n$,

$$|\frac{(-1)^{n-1}x^n}{n}| = \frac{|x|^n}{n} \leq A^n = M_n \;\text{ if } |x| \leq A.$$

Thus, by Thm 71 above, the series for $f(x)$ converges uniformly on $[-A, A]$.
Thus, for $-1 < x \leq 1, \ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots$ pointwise.

20 Power Series

We wish to treat functions like $\sin x, \tan x, \arcsin x, \frac{1}{1+e^x}$, and so on in a similar manner. It is convenient to consider series like $\sum_{n=0}^{\infty} a_n x^n$ in general first. Because the n^{th} term is a power of x those are called 'power series' or 'power series with centre at the origin' or 'about the origin'. A power series with centre $x_0 \in \mathbb{R}$ would be $\sum_{n=0}^{\infty} a_n (x - x_0)^n$. We will develop power series for our important functions later.

Definition The set of $x \in \mathbb{R}$ for which a power series is convergent is called the **interval of convergence of the series**.

e.g. $1 + x + x^2 + x^3 + ...$ converges $\forall x$ with $|x| < 1$ and diverges for x with $|x| > 1$. The interval of convergence is (-1, 1).

e.g. $1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ...$ converges $\forall x$. The interval of convergence is $\mathbb{R}$ itself.

e.g. Every power series, with centre the origin, converges for $x = 0$. However it might not converge at any other point: $1 + x + 2!x^2 + 3!x^3 + ...$ diverges for $x \neq 0$ by the ratio test.

20.1 Radius of Convergence

Let (a_n) be a sequence of real numbers which is bounded, $|a_n| \leq M$ for some number M and all n.

Definition

$$\lim_{n\to\infty} sup a_n = \lim_{n\to\infty} \{sup\{a_n, a_{n+1}, a_{n+2}, ...\}\}.$$

This limit always exists because the sequence $b_n = sup\{a_n, a_{n+1}, a_{n+2}, ...\}$ is monotonically decreasing (prove this!) and bounded below by $-M$.

e.g. $a_n = \frac{1}{n}$ then $limsup a_n = 0$ because $b_n = sup\{\frac{1}{n}, \frac{1}{n+1}, ...\} = \frac{1}{n}$ and $\frac{1}{n} \to 0$.

e.g. $a_n = 1 + (-1)^n, b_n = sup\{2, 0, 2, 0, ...\}$ or $sup\{0, 2, 0, 2, ...\}$ and $b_n = 2$ in each case. Then $limsupa_n = 2$.

Note that $lima_n$ does not exist. However,

Theorem 72 *If* $\lim_{n\to\infty} a_n$ *exists, then* $limsupa_n = lima_n$.

Proof. Let $L = lima_n$. Given $\varepsilon > 0, \exists N_\varepsilon \in \mathbb{N}$ such that for $n \geq N_\varepsilon$,

$$-\varepsilon < L - a_n < \varepsilon, \text{ i.e. } a_n < L + \varepsilon.$$

Thus, if $n \geq N_\varepsilon, b_n \leq L + \varepsilon$ and so $limb_n \leq L + \varepsilon$.
This holds for all $\varepsilon > 0$ and therefore $limb_n \leq L$. Similarly,

$$L - \varepsilon < a_n \ \ \forall n \geq N_\varepsilon \Rightarrow L - \varepsilon \leq b_n \forall n \geq N_\varepsilon \to limb_n \geq L - \varepsilon \Rightarrow limb_n \geq L.$$

Thus $limsupa_n = limb_n = L = l - ma_n$. □

The reason we need to introduce lim sup is the coefficients of the powers of x in a power series form a sequence. If they are bounded in absolute value then lim sup always exists even when their limit does not.
If $f(x) = \sum_{n=0}^{\infty} a_n x^n$ is a power series let $L = limsup \sqrt[n]{|a_n|}$ if the numbers $\sqrt[n]{|a_n|}$ are bounded.
If $L \neq 0$ let the radius of convergence of $f = \frac{1}{L} = \frac{1}{limsup \sqrt[n]{|a_n|}} = R_f$, say.
If $L = 0$, set $R = \infty$.
If the numbers $\sqrt[n]{|a_n|}$ are not bounded set $R = 0$.
For different series all possibilities may exist.

e.g. $\exp x = \sum_{0}^{\infty} \frac{x^n}{n!} : L = limsup \frac{1}{n!}^{\frac{1}{n}} = 0$ because $(n!)^{\frac{1}{n}} \to \infty$.
Thus $R = \infty$.

e.g. $\ln x = x - \frac{x^2}{2} + ...L = limsup(\frac{1}{n})^{\frac{1}{n}} = 1$. Thus $R = \frac{1}{1} = 1$.
These limits are not easy to evaluate and we will give an alternative formula for the radius of convergence which applies in some cases.

Rule If $f(x) = suma_n x^n$ and $|\frac{a_n}{a_{n+1}}| \to$ a limit R then $R = R_f$.

e.g. $f(x) = \ln(1+x) = x - \frac{x^2}{2} + ...|\frac{a_n}{a_{n+1}}| = |\frac{n+1}{n}| \to 1$.
Thus $R_f = 1$.

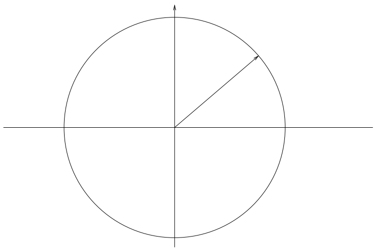
The Rule will be proved later, as will other properties of series related to the radius of convergence. The most important idea is that $f(x)$ converges pointwise when $-R_f < x < R_f$ and uniformly on each interval $[a, b] \subset (-R_f, R_f)$. This latter open interval is called the **interval of convergence**. Lastly, if $|x| > R_f$ then $f(x) = \sum a_n x^n$ **diverges**, in that $a_n x^n$ does not tend to zero when $n = \infty$. If $|x| = R_f$ then the series might converge or diverge. There is always at least one x_0 with $|x_0| = R_f$ and $f(x_0)$ divergent.

e.g. $\ln(1+x) : R_f = 1$, the interval of convergence is (-1, 1).
$\ln(1+1) = 1 - \frac{1}{2} + \frac{1}{3}...$ converges.
$\ln(1-1) = -(1 + \frac{1}{2} + \frac{1}{3}...)$ does not converge.

Note The word "radius" is used because power series are also defined for complex numbers

$$f(z) = \sum_{0}^{\infty} a_n z^n, \quad a_n \in \mathbb{C}, \quad z = x + iy.$$

The theory is the same as that for 'real' series except $|x|$ is replaced by $|z|$ = modulus of $z = \sqrt{x^2 + y^2}$ where $z = x + iy$.



Problems Determine the radius of convergence, interval of convergence, and behaviour of the series at the end points of the interval for the following series:
1. $1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + ...$
2. $10x + 10^2x^2 + 10^3x^3 + ...$
3. $1 + x\cos\theta + x^2\cos 2\theta + ...$
4. $x + 2!x^2 + 3!x^3 + ...$

5. $1 + mx + \frac{m(m-1)x^2}{2!} + \frac{m(m-1)(m-2)x^3}{3!} + ...,\ m \in \mathbb{R}$.
6. $x + \frac{1}{2} \cdot \frac{x^3}{3} + \frac{1}{4} \cdot \frac{3}{5} \cdot \frac{x^5}{5} + ...$
7. $1 - 2x + 3x^2 - 4x^3 + ...$
8. $\frac{1}{a} + \frac{bx}{a^2} + \frac{b^2x^2}{a^3} + \frac{b^3x^3}{a^4} + ...,\ a, b \in \mathbb{R}, a \neq 0$.

The following theorem was first proved by Abel. It gives valuable insight into the convergence properties of power series:

Theorem 73 *If $f(x) = \sum a_n x^n$ is convergent for $x = x_0$, then it is absolutely convergent for all x satisfying $|x| < |x_0|$. On the other hand, if the series is divergent for $x = x_0$ then it is divergent $\forall x$ with $|x| > |x_0|$.*

Proof. Let f converge for $x = x_0 \neq 0$. Then $a_n x_0^n \to 0$. Therefore there exists a $B \geq 0$ such that $|a_n x_0^n| \leq B\ \ \forall n \in \mathbb{N}$.
Then

$$|a_n x^n| = |a_n x_0^n \cdot \frac{x^n}{x_0^n}| = |a_n x_0^n||\frac{x}{x_0}|^n \leq B|\frac{x}{x_0}|^n.$$

If $|x| < |x_0|$ then $|\frac{x}{x_0}| = a < 1$. Therefore $|a_n x^n| \leq Ba^n\ \ \forall n$ and $0 \leq a < 1$. The series $\sum_{n=1}^{\infty} Ba^n$ converges. Therefore, the series $\sum a_n x^n$, by the comparison test, converges absolutely for this value of x.
If f diverges for $x = x_0$ and converges for some x with $|x| > |x_0|$, then, by what we have just proved, f converges at x_0 since $|x_0| < |x|$. This is a contradiction and proves the second part of the theorem. □

Second definition of the radius of convergence
Let $f(x) = \sum a_n x^n$ be a power series. Then the radius of convergence R_f of f is the unique number satisfying the following property: f converges absolutely when $|x| < R_f$ and diverges when $|x| > R_f$.

Theorem 74 *The two definitions of radius of convergence are the same.*

Proof. Let $L = limsup \sqrt[n]{|a_n|}$.
Case 1: $L = 0$ and $R = \infty$. Then let $x \neq 0 \Rightarrow \frac{1}{2|x|} > 0$. Let $\varepsilon = \frac{1}{2|x|}$. Because $0 = limsup \sqrt[n]{|a_n|}$ it follows that at most a finite number of the $\sqrt[n]{|a_n|} \geq \varepsilon$. If not, if $b_n = sup\{ \sqrt[n]{|a_n|}, \sqrt[n+1]{|a_{n+1}|}, ...\}$ we would have $b_n \geq$

ε $\forall n$ thus *limsup* $\sqrt[n]{|a_n|} \geq \varepsilon > 0$, a contradiction. Therefore, $\exists N_\varepsilon$ such that $\forall n \geq N_\varepsilon$,

$$\sqrt[n]{|a_n|} < \frac{1}{2|x|} = \varepsilon.$$

This implies $|a_n x^n| < \frac{1}{2^n}$, and so, by the comparison test, $f(x)$ converges absolutely for $x \neq 0$. If $x = 0$ this result is also true.

Case 2: $0 < L < \infty$.
Let x be any fixed number satisfying $|x| < R_f = \frac{1}{L}$.
Choose a number r satisfying $|x| < r < \frac{1}{L}$ and let $\varepsilon = \frac{1}{r} - L > 0$. Only a finite number of the $\sqrt[n]{|a_n|}$ can exceed $L + \varepsilon$ (otherwise we would get a contradiction as in Case 1 above).
Thus $\exists N_\varepsilon \in \mathbb{N}$ such that $\forall n \geq N_\varepsilon$, $\sqrt[n]{|a_n|} \leq L + \varepsilon = \frac{1}{r}$.
Thus

$$|a_n x^n| \leq \frac{|x|^n}{r^n} < 1 \text{ as } |x| < r.$$

Therefore, by the comparison test, $\sum a_n x^n$ converges absolutely for this value of x.
If $|x| > R_f = \frac{1}{L}$ then $|a_n x^n| > 1$ for an infinite number of integers n (you should prove this exercise) and therefore the series cannot converge for these values of x. □

Proof of the rule: If the series $f(x) = \sum a_n x^n$ is such that the limit $\lim_{n\to\infty} |\frac{a_n}{a_{n+1}}|$ exists then the value of this limit is the radius of convergence R_f.

Proof. By D'Alembert's test, the series f will converge absolutely for values of x satisfying

$$\lim_{n\to\infty} |\frac{a_{n+1}x^{n+1}}{a_n x^n}| < 1 \Leftrightarrow |x| < \frac{1}{\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}|},$$

and it will diverge if

$$|x| > \frac{1}{\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}|}.$$

The rule follows from the second definition of the radius of convergence. □

Theorem 75 *Let R_f be the radius of convergence of the power series*

$$f(x) = \sum a_n x^n.$$

Then f converges uniformly on every closed and bounded interval $[a,b] \in (\mathbb{R}_f, R_f)$.

Proof. If $R_f = 0$ the result is easy.
If $R_f \neq 0$ then there is a number $A > 0$ with $[a,b] \in [-A, A] \in (-R_f, R_f)$. Because $|A| < R_f$ the series converges absolutely at $x = A$. If $|x| \leq A$ then $|a_n x^n| \leq |a_n A^n| = M_n$, say. Note that $\sum M_n < \infty$ because of what we have just said. By the Wierstrass M-test, f converges uniformly for $|x| \leq A$. □

Theorem 76 *A power series f converges to a continuous function on the open interval $(-R_f, R_f)$.*

Proof. Every $a_n x^n$ is a continuous function. The result then follows from Thm 71 above. □

Theorem 77 *If R_f is the radius of convergence of the power series f then f is differentiable on $(-R_f, R_f)$, the derivative has a convergent power series given by $f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$, and the radius of convergence of the power series is also R_f.*

Proof. The last statement follows from the fact that

$$\lim_{n\to\infty} sup \sqrt[n]{|na_n|} = limsup \sqrt[n]{n} \cdot \sqrt[n]{a_n} = \lim \sqrt[n]{n} \cdot limsup \sqrt[n]{|a_n|} = 1 \cdot limsup \sqrt[n]{|a_n|}.$$

The above statement says that functions defined by convergent power series are differentiable at all points within the circle of convergence. We will prove this using Thm 64 and 75.

Let $f_n(x) = a_n x^n$. Then $f_n \in C^{(1)}[a,b]$ if $[a,b] \subset (-R_f, R_f)$. By Thm 75, f_n' converges on $[a,b]$. Thus, by Thm 64,

$$\sum f_n'(x) = f'(x) \text{ or} \sum (a_n x^n)' = \sum n a_n x^{n-1}.$$

□

e.g. If $[a, b] \subset (-R_f, R_f)$ then

$$\int_a^b \sum a_n x^n dx = \sum (\frac{a_n}{n+1} x^{n+1})|_a^b.$$

Theorem 78 Uniqueness
If the power series $\sum a_n x^n$ and $\sum b_n x^n$ both converge in some open interval about the point $x = 0$ and have the same sum for every value of x in this interval, then $a_n = b_n$ $\forall n = 0, 1, 2, ...$

Proof. $\exists r > 0$ such that

$$a_0 + a_1 x + a_2 x^2 + ... = b_0 + b_1 x + b_2 x^2 + ... \quad \forall x \in (-r, r).$$

Let $x = 0 \Rightarrow a_0 + 0 = b_0 + 0 \Rightarrow a_0 + b_0$.
By Thm 77,

$$(a_0 + a_1 x + a_2 x^2 + ...)' = (b_0 + b_1 x + b_2 x^2 + ...)'$$

$$\Rightarrow a_1 + 2a_2 x + ... = b_1 + 2b_2 x + ...$$

Let $x = 0 \Rightarrow a_b = b_1$.
Similarly, $(\sum a_i x^i)^{(n)} = (\sum b_i x^i)^{(n)}$

$$\Rightarrow a_n = b_n.$$

□

Theorem 79 Cauchy Product of Power Series
If $f(x) = \sum a_n x^n$ and $g(x) = \sum b_n x^n$ have radii of convergence R_f and R_g respectively, define the Cauchy Product of f and g by

$$(fg)(x) = \sum_{n=0}^{\infty} (\sum_{l=m=n} a_l b_m) x^n.$$

Then $(fg)(x) = f(x)g(x)$ at least in $(-R_f, R_f) \bigcap (-R_g, R_g)$.

Proof. This follows from Thm 48 on the multiplication of series and the fact that power series converges absolutely at every x in its interval of convergence. □

Exercise Write down a power series expansion for $\frac{e^x}{1-x}$ and determine its interval of convergence.

21 $\sin x$ and $\cos x$

Let

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - ... + \frac{(-1)^n x^{2n+1}}{(2n+1)!} + ...$$

and

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - ... + \frac{(-1)^n x^{2n}}{(2n!)} + ...$$

Then $R_{sin} = \frac{1}{limsup \frac{1}{(2n!)}^{2n}} = \infty = R_{cos}$.
Thus, by Thm 75, the two series converge uniformly on every closed and bounded interval $[a, b] \subset \mathbb{R}$: therefore, by an earlier theorem on the continuity of polynomials, sin and cos are continuous on $\mathbb{R}$. By Thm 77:

$$(\sin x)' = (x - \frac{x^3}{3!} + ...)' = 1 - \frac{x^2}{2!} + ... = \cos x.$$

Similarly, $(\cos x)' = -\sin x$ by Thm 77.

21.1 Properties of sin and cos

1.

$$\begin{aligned}\sin(-x) &= \sum_{n=0}^{\infty} \frac{(-1)^n (-x)^{2n+1}}{(2n+1)!} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^{2n+1+n} x^{2n+1}}{(2n+1)!} \\ &= -\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \\ &= -\sin x.\end{aligned}$$

2. $\cos(-x) = \cos x$ because $(-1)^{2n} = 1 \ \ \forall n \in \mathbb{Z}$.

3. $\sin(0) = \sum \frac{(-1)^n 0^{2n+1}}{(2n+1)!} = 0$.

4. $\cos(0) = 1 - 0 = 1$.

5. $\sin^2 x + \cos^2 x = 1$. To see this, let

$$f(x) = \sin^2 x + \cos^2 x.$$

Then

$$f'(x) = 2\sin x \cos x - 2\cos x \sin x = 0 \quad \forall x \in \mathbb{R}.$$

Therefore, $f(x) = const. = f(0) = 0^2 + 1^2$ by 3. and 4. Hence $\sin^2 x + \cos^2 x = 1$.

6.

$$\cos(x+y) = \cos x \cos y - \sin x \sin y$$

Because the power series for $\sin x$ and $\cos x, \sin y$ and $\cos y$ converge absolutely, we can multiply the series together using the Cauchy product and then add the resulting terms together in any order:

RHS $=$ $\cos x \cos y - \sin x \sin y$

$$= (1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots)\cdot(1 - \frac{y^2}{2!} + \frac{y^4}{4!} - \ldots) - (x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots)\cdot(y - \frac{y^3}{3!} + \frac{y^5}{5!} - \ldots)$$

$$= (1 - (\tfrac{x^2}{2!} + xy + \tfrac{y^2}{2!}) + (\tfrac{x^4}{4!} + \tfrac{x^3}{3!}y + \tfrac{x^2}{2!}\tfrac{y^2}{2!} + \tfrac{y^3}{3!}x + \tfrac{y^4}{4!}\ldots)$$

$$= 1 - \tfrac{(x+y)^2}{2!} + \tfrac{(x+y)^4}{4!} - \ldots = \cos(x+y) = \text{ LHS}.$$

Note It is much easier to use complex series and define $\exp z = \sum_0^\infty \frac{z^n}{n!}$. First prove $\exp(z_1+z_2) = \exp(z_1)\exp(z_2)$ using the same method used in the proof of Theorem 65. If $z = x + iy$, one would then show

$$\exp z = e^x(\cos y + i \sin y).$$

The expression for $\cos(x+y)$ will then follow from this without having to multiply series.

7. $\sin(x+y) = \sin x \cos y + \cos x \sin y$:
Let $a \in \mathbb{R}$ be fixed and let

$$f(x) = \sin(x+a) - \sin x \cos a - \sin x \cos a - \cos x \sin a.$$

Then

$$f'(x) = \cos(x + a) - \cos x \cos a + \sin x \sin a = 0$$

by 6. Thus

$$f(x) = const. = f(-a) = \sin(-a + a) - s \in (-a) \cos(a) - \cos(-a) \sin(a) = 0$$

by 1. and 2.

Therefore,

$$0 = \sin(x + a) - \sin x \cos a - \cos x \sin a$$

and the result follows by letting $a = y$.

8. By 5., $\sin^2 x \leq 1$ and $\cos^2 x \leq 1$.
Hence $-1 \leq \sin x \leq 1$ and $-1 \leq \cos x \leq 1$.

9. **Definition of the real number** π
Let

$$S = \{x \in \mathbb{R} : x > 0 \text{ and} \cos x - 1\}.$$

If $\exists \beta > 0$ such that $\cos \beta = -1$, then

$$\cos 2\beta = 2\cos^2 \beta - 1 = 1 \Rightarrow 2\beta \in S.$$

If $\exists \gamma > 0$ such that $\cos \gamma = 0$, then

$$\cos 2\gamma = 2\cos^2 \gamma - 1 = -1 \Rightarrow \cos(4\gamma) = 1 \text{ and so } 4\gamma \in S.$$

Since $\cos 0 = 1$, either a $\gamma > 0 \ \exists$ with $\cos \gamma = 0$ or, by the Intermediate Value theorem and the continuity of $\cos x$, $\cos x > 0 \ \forall x \geq 0$.

Suppose $\cos x > 0 \ \forall x \geq 0$ (??). Then $\sin'(x) = \cos x > 0 \ \forall x > 0$.

Thus $\sin x$ is strictly monotonically increasing on $(0, \infty)$ and bounded above by 1.

Thus $\lim\limits_{x \to \infty} \sin x = m \exists$ and $0 < m \leq 1$.

Also, $\cos'(x) = -\sin x < 0$ on $(0, \infty)$ implies $\cos x$ is monotonically decreasing and bounded below by 0.

Thus, $\lim_{x\to\infty} \cos x = l \ \exists$ and $0 \le l < 1$ as $\cos 0 = 1$.
But

$$\sin 2x = 2\sin x \cos x \Rightarrow \lim_{x\to\infty} \sin 2x = m = 2 \lim_{x\to\infty} \sin x \cdot \cos x = 2ml = \frac{1}{2}$$

and

$$\cos 2x = 2\cos^2 x - 1 \Rightarrow l = 2l^2 - 1 \ \Rightarrow \frac{1}{2} = -\frac{1}{2}(!!)$$

Therefore there must be at least one $\gamma > 0$ with $\cos\gamma = 0$ and thus an $\alpha < 0$ with $\cos\alpha = 1$. Hence $S \neq 0$.

Let $2\pi = glb(x)$ (defines π).
Then $\exists \alpha_n \in S$ such that $\lim_{n\to\infty} \alpha_n = 2\pi$ and $\cos\alpha_n = 1$.
Hence

$$1 = \lim_{n\to\infty} \cos(\alpha_n)$$
$$= \cos(\lim_{n\to\infty} \alpha_n)$$

since cos is continuous

$$= \cos(2\pi).$$

Therefore $2\pi \in S$ if $2\pi \neq 0$.
By Taylor's Theorem (with remainder of order 4),

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!}\cos(\theta x) \quad 0 < \theta < 1$$

$$\le 1 - \frac{x^2}{2!} + \frac{x^4}{4!} \ \text{ since } \ \cos(\theta x) \le 1$$

< 1 if $0 < x < 1$.
Therefore, $2\pi \ge 1$ and so $\pi \neq 0$.

10.

$$\cos(2\pi + x) = \cos x \ \ \forall x \in \mathbb{R}$$

$$\cos^2(2\pi) + \sin^2(2\pi) = 1 \Rightarrow \sin(2\pi) = 0$$

$$\cos(2\pi + x) = \cos x \cos(2\pi) - \sin x \sin(2\pi)$$

$$= \cos x \cdot 1 - \sin x \cdot 0 = \cos x$$

22 Binomial Theorem

Let $m \in \mathbb{R}$ and $x \in \mathbb{R}$. Then

$$f(x) = (1+x)^m = 1 + mx + \frac{m(m-1)x^2}{1 \cdot 2} + \frac{m(m-1)(m-2)x^3}{1 \cdot 2 \cdot 3} + ...$$

$$= \sum_{r=0}^{\infty} \begin{pmatrix} m \\ r \end{pmatrix} x^r$$

where

$$\begin{pmatrix} m \\ r \end{pmatrix} = m(m-1)(m-2)...\frac{(m-r+1)}{r!} \quad \forall m, \forall r = 0, 1, 2, ...$$

The convergence is uniform on $\mathbb{R}$ if $m = 0, 1, 2, \ldots$. In this case the series terminates after a finite number of terms as $\begin{pmatrix} m \\ r \end{pmatrix} = 0$ when $r > m$ (since the term $(m-(m+1)+1)$ in the numerator is zero). The theorem is then the usual Binomial Theorem. If $m \neq 0, 1, 2, ...$ then the series **never** terminates and the radius of convergence is 1.

Proof. If $m = 0$ then $(1+x)^m = 1 = 1 + 0$ and we are done.

If $m = 1, 2, 3, ...$ then $(1+x)^m = \sum_{r=0}^{m} \begin{pmatrix} m \\ r \end{pmatrix} x^r$. This is proved by induction on m, and, you will recall, the proof depends on properties of the binomial coefficients. It is important not to use Taylor's theorem as this may involve circularity. If $m \neq 0, 1, 2, ...$ use Taylor's theorem to find the series (using $(x^\alpha)' = \alpha x^{\alpha-1}$ for $\alpha \in \mathbb{R}$) and then test the series for uniform convergence (after checking that the remainder $\to 0$). Because the series is a power series, we can find its radius of convergence. However, we still need to check that it converges to $(1+x)^m$ and not some other function within its circle of convergence. □

Lemma If $a \in \mathbb{R}$ and $x \in (0, \infty)$ and $f(x) = x^a$, then $f'(x) = ax^{a-1}$.

Proof.

$$\ln f(x) = a \ln x \Rightarrow \frac{f'(x)}{f(x)} = \frac{a}{x} \Rightarrow f'(x) = \frac{ax^a}{x} = ax^{a-1}.$$

If $f(x) = (1+x)^m$ and $-1 < x$ then, by the lemma,

$$f^{(1)}(x) = m(1+x)^{m-1}$$

and

$$f^{(n)}(x) = m(m-1)(...)(m-n+1)(1+x)^{m-n}.$$

Thus

$$f(x) = f(0) + \frac{xf'(0)}{1!} + ... + \frac{f^{n-1}(0)x^{n-1}}{(n-1)!} + R_n(x, \xi)$$

where $0 < \xi < x$ or $x < \xi < 0$ and $R_n = \frac{f^n(\xi)x^n}{n!}$.
Thus

$$f(x) = (1+x)^m = 1 + mx + ... + m(m-1)(..)\frac{(m-n+2)x^{n-1}}{(n-1)!} + R_n$$

where

$$R_n = \frac{m(m-1)...(m-n+1)(1+\xi)^{m-n}x^n}{n!}.$$

If $a_n = \frac{f^n(0)}{n!}$ then $|\frac{a_n}{a_{n+1}}| = |\frac{n+1}{m-n}| = |\frac{1+\frac{1}{n}}{1-\frac{m}{n}}| \to 1$.
Thus $R_f = 1$. □

Exercise Prove that $R_n(x, \xi) \to 0$ when $0 < \xi < x < 1$.

In order to prove that $f(x) = \sum_0^\infty a_n x^n$ pointwise in (-1, 1) we need to develop an alternative form for the remainder in Taylor's theorem.

Theorem 80 Taylor's Theorem - Second Form
Let $f \in C^{(n)}(\alpha, \beta)$ and let $\{a, a+h\} \in (\alpha, \beta)$. Then

$$f(a+h) = f(a) + \frac{hf'(a)}{1!} + ... + \frac{h^{n-1}f^{(n-1)}(a)}{(n-1)!} + R_n$$

where

$$R_n = \frac{h^n}{(n-1)!} \cdot \int_0^1 (1-t)^{n-1} f^{(n)}(a+th)dt.$$

If $p \in \{1, 2, ..., n\}$ there exists a θ with $0 < \theta < 1$ (θ depends in general on $a, h, p,$ and n) such that

$$R_n = \frac{(1-\theta)^{n-p} f^{(n)}(a+\theta h)h^n}{p(n-1)!}$$

Note
i. If $p = n$ *we obtain the first form for the remainder (Lagrange form).*
ii. The first form requires $f^{(n)}(x)$ *to exist but this form requires that it also be continuous.*
iii. If $p = 1$ *we obtain Cauchy's form for* R_n*, namely*

$$R_n = \frac{(1-\theta)^{(n-1)} f^{(n)}(a+\theta h)}{(n-1)!}$$

This is the form we will use to complete the proof of the Binomial Theorem.

Proof. Let $x, b \in (\alpha, \beta)$ and let

$$F_n(x) = f(b) - f(x) - (b-x)f'(x) - ... - \frac{(b-x)^{n-1}}{(n-1)!} f^{(n-1)}(x)$$

Then

$$F_n'(x) = -\frac{(b-x)^{n-1}}{(n-1)!} f^{(n)}(x)$$

and therefore

$$F_n(a) = F_n(b) - \int_a^b F_n'(x)dx = \frac{1}{(n-1)!}\int_a^b (b-x)^{n-1} f^{(n)}(x)\, dx.$$

If now we let $b = a + h$ and transform the integral by setting $x = a + th$, (a, h const, t variable) we obtain

$$f(a+h) = f(a) + hf'(a) + ... + \frac{h^{n-1} f^{(n-1)}(a)}{(n-1)!} + R_n$$

where

$$R_n = \frac{h^n}{(n-1)!}\int_0^1 (1-t)^{n-1} f^{(n)}(a+th)dt.$$

This is called, sometimes, the integral form for the remainder. Note that we needed some hypothesis on $f^{(n)}(x)$ to ensure that the integral exists. Now let $p \in \{1, ..., n\}$ be given. Write

$$R_n = \frac{h^n}{(n-1)!}\int_0^1 (1-t)^{n-p}(1-t)^{p-1} f^{(n)}(a+th)at. \quad *$$

□

Lemma If f and ϕ are continuous on $[a,b]$ and $\phi \neq 0$, then $\exists \xi$ with $a < \xi < b$ and

$$\int_a^b f(x)\phi(x)dx = f(\xi)\int_a^b \phi(x)dx$$

(a very useful mean value theorem for integrals).

Proof of the lemma Let

$$F(x) = \int_a^x f(u)\phi(u)du \text{ and } G(x) = \int_a^x \phi(u)du$$

and apply the method used in the proof of Thm 48 on indeterminate forms. At the beginning of the proof of this theorem a formula which implies $(n = 1, F(a) = G(a) = 0)$

$$\frac{F(b)}{G(b)} = \frac{F\text{"}(\xi)}{G'(\xi)}.$$

But $F'(\xi) = f(\xi)\phi(\xi)$ and $G'(\xi) = \phi(\xi)$. Thus

$$\frac{F(b)}{G(b)} = \frac{f(\xi)\phi(\xi)}{\phi(\xi)} = f(\xi)$$

and so

$$F(b) = f(\xi)G(b) = f(\xi)\int_a^b \phi(u)du.$$

Apply the result of the above lemma to the formula for R_n given by $*$: $\exists\theta$ with $0 < \theta < 1$ such that

$$R_n = \frac{h^n}{(n-1)!}(1-\theta)^{n-p}f^{(n)}(a+\theta h)\int_0^1 (1-t)^{p-1}\,dt$$

$$= \frac{(1-\theta)^{n-p}f^{(n)}(a+\theta h)h^n}{p(n-1)!}.$$

We now return to the Binomial Theorem where $f(x) = (1+x)^m$. Let $p = 1$. Then $a = 0, h = x$, and

$$R_n = \frac{m(m-1)(...)(m-n+1)}{1 \cdot 2...(n-1)}\frac{(1-\theta)^{n-1}x^n}{(1+\theta x)^{n-m}}$$

when $-1 < x < 1$ we have $-1 < x$ and so $-\theta < \theta x \Rightarrow 1-\theta < 1+\theta x \Rightarrow 0 < \frac{1-\theta}{1+\theta x} < 1$.

If $m > 1$ then $(1+\theta x)^{m-1} < (1+|x|)^{m-1}$ and
if $m < 1$ then $(1+\theta x)^{m-1} < (1-|x|)^{m-1}$ hence

$$|R_n| < |m|(1 \pm |x|)^{m-1}| \begin{pmatrix} m-1 \\ n-1 \end{pmatrix} ||x|^n \, \alpha_n$$

independent of n .
To show $\alpha_n \to 0$ when $n \to \infty$ consider the power series $\sum \begin{pmatrix} m \\ n \end{pmatrix} x^n$. Its radius of convergence is 1 and so the n^{th} terms goes to zero when $-1 < x < 1$, i.e. $|\begin{pmatrix} m \\ n \end{pmatrix} x^n| \to 0 \Rightarrow |\begin{pmatrix} m-1 \\ n-1 \end{pmatrix}||x^n| \to 0$. The proof of the Binomial Theorem is now complete.

Corollary If $a \neq 0, b \neq 0$ then

$$(a+bx)^m = a^m(1+\frac{bx}{a})^m = a^m \sum_{n=0}^{\infty} \begin{pmatrix} m \\ n \end{pmatrix} \frac{b^n x^n}{a^n}$$

converges pointwise when $-1 < \frac{bx}{a} < 1 \Leftrightarrow |x| < |\frac{a}{b}|$.

Exercise Use the Binomial Theorem to prove

$$\frac{1}{1-x} = 1 + x + x^2 + ...$$

when $|x| > 1$.

23 arctan and tan

The following expansions are valid within some interval of convergence. They could be developed using Taylor series about $x = 0$ in each case.
a.

$$\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} + ...$$

b.

$$\sec x = 1 + \frac{x^2}{2!} + \frac{5x^4}{4!} + \frac{61x^6}{6!} + ...$$

c.

$$\arcsin x = x + \frac{1}{2}\frac{x^3}{3} + \frac{1}{2}\frac{3}{4}\frac{x^5}{5} + \frac{1}{2}\frac{3}{4}\frac{5}{6}\frac{x^7}{7} + ...$$

d.

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + ...$$

However, we will use other methods to derive some of them.

a. The series for $\tan x = \frac{\sin x}{\cos x} = \sin x \cdot \sec x$ can be found by the division or manipulation of series as described below.

b.

$$\sec x = \frac{1}{\cos x} = \frac{1}{1 - \frac{x^2}{2!} + \frac{x^4}{4!} - ...}$$

$$= \frac{1}{1-z} \text{ where } z = \frac{x^2}{2!} - \frac{x^4}{4!} + ...$$

$$= 1 + z + z^2 + ... \text{ if } |z| < 1$$

$$= 1 + (\frac{x^2}{2!} - \frac{x^4}{4!} + ...) + (\frac{x^2}{2!} - \frac{x^4}{4!} + ...)^2 + ...$$

$$= 1 + \frac{x^2}{2!} + x^4(-\frac{1}{4!} + (\frac{1}{2!})^2) + ...$$

$= 1 + \frac{x^2}{2!} + \frac{5x^4}{4!} + ...$

The process is not rigorous! It certainly doesn't work for x with $\cos x = 0$. We cannot prove a theorem about the inversion ($\frac{1}{a\ powerseries}$ = a power series) of power series in this course - instead we will give another method of performing the inversion:

Let $\frac{1}{\cos x} = \sum_{n=0}^{\infty} a_n x^n$ where the numbers (a_n) have yet to be determined. Then

$$1 = \cos x \sum_{n=0}^{\infty} a_n x^n = (1 - \frac{x^2}{2!} + \frac{x^4}{4!} - ...)(a_0 + a_1 x + a_2 x^2 + ...)$$

$$\Rightarrow 1 = a_0 + a_1 x + x^2(-\frac{a_0}{2} + a_2) + ...$$

The values of the numbers (a_n) can then be determined by equating the coefficients of the different powers of x on each side:

$$1 + a_0 \Rightarrow a_0 = 1$$

$$0 = a_1 \Rightarrow a_1 = 0$$

$$0 = -\frac{a_0}{2} + a_2 \Rightarrow a_2 = \frac{a_0}{2} = \frac{1}{2}.$$

Thus $\sec x = 1 + \frac{x^2}{2} + ...$ as before.

c. If $f^-1(x) = \sin x$ then $f(x) = \arcsin x$ and $f'(x) = \frac{1}{\sqrt{1-x^2}} = (1-x^2)^{-\frac{1}{2}}$ if $|x| < 1$.
Replace x by $(-x^2)$ in the Binomial Theorem and m by $-\frac{1}{2}$. We obtain a series valid for $-1 < -x^2 < 1 \Leftrightarrow x^2 < 1 \Leftrightarrow -1 < x < 1$. The series is

$$f'(x) = (1-x^2)^{-}\frac{1}{2} = \sum_{n=0}^{\infty} \frac{(-1)^n 1.3.5...(2n-1)x^{2n}(-1)^n}{n!2^n}$$

$$= \sum_{n=0}^{\infty} \frac{1.2.3...(2n-1)x^{2n}}{n!2^n}$$

Integrate from 0 to x: this is possible when

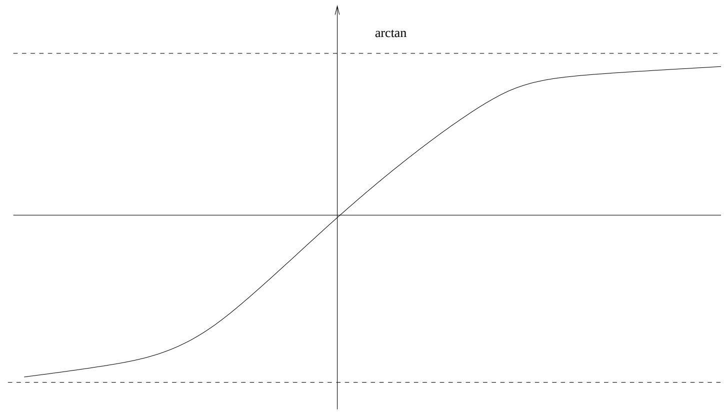
$$-R_f < x < R_f \quad \text{since} \quad [0, x] \subset (-R_f, R - f)$$

and convergence is uniform on this interval. We obtain

$$f(x) = \arcsin x = \int_0^x f'(u)du = \sum_{n=0}^{\infty} \frac{1.2.3...(2n-1)x^{2n+1}}{2^n n!(2n+1)}.$$

The reader should check the first three terms to see that this is the series given in c.

d. The same method can be used to derive the series for arctan using $f(x) = \arctan \Rightarrow f'(x) = (1+x^2)^-1$. Expand using the Binomial Theorem and then integrate term by term. Observe that the given series for arctan has a radius of converge $R_f = 1$ and does converge at $x = 1$. However the domain of arctan is the whole of $\mathbb{R}$.



arctan cannot be represented by a series which converges on the whole of $\mathbb{R}$.