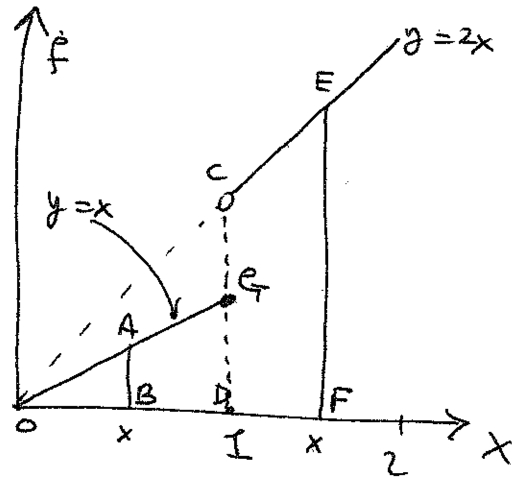
#4. $f(x) = \begin{cases} x, & x \le 1 \\ 2x, & x > 1. \end{cases}$



If $0 \le x \le 1$

$F(x) = \int_0^x f = \text{area } \Delta OAB = \frac{1}{2} x . x = \frac{x^2}{2}$

If $1 \le x \le 2$

$F(x) = \int_0^x f =$ area ΔOGD + area trapezium $DCEF$.

$$= \frac{1}{2} + \frac{1}{2}(|DC| + |EF|)(x-1)$$

$$= \frac{1}{2} + \frac{1}{2}(2 + 2x)(x-1)$$

$$= \frac{1}{2} + (1+x)(x-1) = \frac{1}{2} + x^2 - 1 = x^2 - \frac{1}{2}$$

$\lim_{x \to 1-} F(x) = \lim_{x \to 1-} \frac{x^2}{2} = \lim_{x \to 1} \frac{x^2}{2} = \frac{1^2}{2} = \frac{1}{2} = F(1)$

$+ \lim_{x \to 1+} F(x) = \lim_{x \to 1+} (x^2 - \frac{1}{2}) = 1 - \frac{1}{2} = \frac{1}{2}$

$\therefore \lim_{x \to 1} F(x) = \frac{1}{2} = F(1)$ so F is continuous at $x = 1$.

If $0 < x < 1$ $F'(x) = (\frac{x^2}{2})' = x = f(x)$.

If $1 < x < 2$ $F'(x) = (x^2 - \frac{1}{2})' = 2x = f(x)$

$\therefore$ if $x \ne 1$ and $x \in (0,2)$, $F'(x) = f(x)$.

$F'(1)$ does not exist, since

$F'_-(1) = 1$ and $F'_+(1) = 2$

#5 $\varepsilon_0 = 1$. Let $\delta = \frac{1}{n}$ and $x_n = \frac{1}{n}$ and $y_n = \frac{1}{n+1}$.

Then $x_n, y_n \in (0,1]$ and $|x_n - y_n| = \frac{1}{n} - \frac{1}{n+1} = \frac{n+1-n}{n(n+1)} = \frac{1}{n(n+1)} < \delta$

But $|f(x_n) - f(y_n)| = \left| \frac{1}{1/n} - \frac{1}{1/(n+1)} \right| = |n - (n+1)| = 1 \geq \varepsilon_0$.

Hence $\exists\, \varepsilon > 0$ such that no matter how small we choose $\delta > 0$, there are always points $x, y \in (0,1]$ with $|x-y| < \delta$ but $|f(x) - f(y)| \geq \varepsilon$.

Hence $f(x) = 1/x$ is not uniformly continuous on $(0,1]$ //

#6 Let f, g be Riemann integrable on $[a,b]$.

① $\frac{1}{4}\left((f+g)^2 - (f-g)^2\right) = \frac{1}{4}\left(f^2 + 2fg + g^2 - (f^2 - 2fg + g^2)\right)$

$$= \frac{4fg}{4} = fg$$

② Hence since $f \pm g$ are Riemann integrable, if we can show the square of any Riemann integrable function is Riemann integrable then we would be done, since $f \cdot g$ is expressed in terms of squares.

③ Since $f^2(x) = |f(x)|^2$ and f Ri Int $\Rightarrow |f|$ is Ri Int, we can replace $f(x)$ by $|f(x)|$ and so need only show the square of a <u>positive</u> Riemann integrable function is Riemann integrable.

④ So let $f(x) \geq 0$ and assume $0 \leq f(x) \leq M$, $M \in \mathbb{R}$.

Then $x, y \in [a,b] \Rightarrow f^2(x) - f^2(y) \overset{\boxtimes}{=} (f(x) + f(y))(f(x) - f(y))$

So if ~~$\S$~~ $P = (x_0, \dots, x_n)$ is any partition of $[a,b]$ and, with the usual notation

$M_i = \sup\{f(x) : x \in I_i\}$ $\qquad m_i = \inf\{f(x) : x \in I_i\}$

$M_i' = \sup\{f^2(x) : x \in I_i\}$ $\qquad m_i' = \inf\{f^2(x) : x \in I_i\}$.

We can find an x so $\quad M_i' - \varepsilon < f^2(x)$

and a y so $\quad f^2(y) < m_i' + \varepsilon$

$\Rightarrow M_i' - m_i' - 2\varepsilon < f^2(x) - f^2(y)$ $\boxtimes$

$\boxtimes \Rightarrow M_i' - m_i' - 2\varepsilon < (f(x) + f(y))(f(x) - f(y)) \leq 2M(f(x) - f(y))$

$$\leq 2M(M_i - m_i)$$

But this is true for all $\varepsilon > 0$ $\therefore$

$$M_i' - m_i' \leq 2M(M_i - m_i)$$

$\times$ by Δx_i and sum to get $\quad U(f^2, P) - L(f^2, P) \leq 2M(U(f,P) - L(f,P))$

Since f is Ri Int we can make the RHS small, hence the LHS small & thus, by Thm 50 $f^2(x)$ (& thus $f \cdot g$) is Ri Int //