(4) $f(x) = x(x-1)(x-3) = x(x^2-4x+3) \overset{\boxtimes}{=} 3x - 4x^2 + x^3$

$\Rightarrow f'(x) = 3 - 8x + 3x^2$

$f''(x) = -8 + 6x$

$\Rightarrow f(0+h) = f(0) + f'(0)h + \frac{f''(0+\theta h)}{2!}h^2$

$\boxtimes \Rightarrow 3h - 4h^2 + h^3 = 0 + 3h + \frac{1}{2}(6\theta h - 8)h^2$

$\Rightarrow -4h^2 + h^3 = (3\theta h - 4)h^2$

$\Rightarrow -4h^2 + h^3 = 3\theta h^3 - 4h^2 \Rightarrow h^3 = 3\theta h^3 \Rightarrow \boxed{\theta = \frac{1}{3}}$

& $\frac{1}{3} = |\frac{1}{3}| < 1$.

---

(5) $f^{(4)}(x) = (6)' = 0$ & $f'''(x) = 6 \quad \forall x \in \mathbb{R}$.

$\therefore f^{(n)}(x) = 0 \quad \forall n \geq 4$ so we can write

$f(a+h) = f(a) + f'(a)h + \frac{f''(a)}{2!}h^2 + \frac{1}{3!}f'''(a)h^3 + 0 \quad \forall a \in \mathbb{R}$.

This is just via $\boxtimes$

$3(a+h) - 4(a+h)^2 + (a+h)^3 =$

$(3a - 4a^2 + a^3) + (3 - 8a + 3a^2)h + \frac{1}{2}(-8+6a)h^2 + \frac{1}{6}(6)h^3$

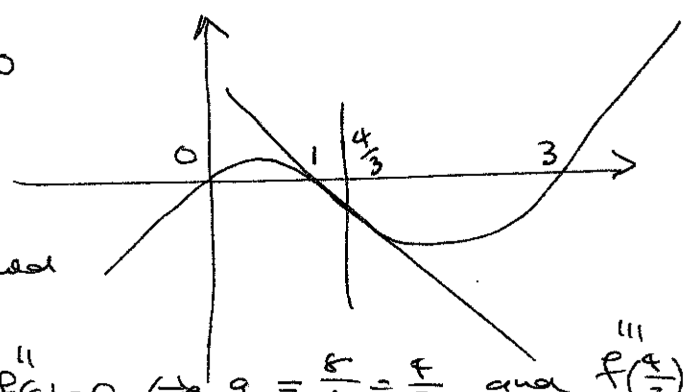
---

(6) A pt. of inflection is where the tangent crosses the curve,

i.e. $f''(a) = 0$, $f'''(a) = 0$, ... $f^{(2n)}(a) = 0$

$f^{(2n+1)}(a) \neq 0$

i.e. other than the first, the first unvanishing derivative should be of odd order.



Here $f''(x) = 6x - 8$ so $f''(a) = 0 \Leftrightarrow a = \frac{8}{6} = \frac{4}{3}$ and $f'''(\frac{4}{3}) = 6 \neq 0$

$\therefore x = 4/3$ is a pt. of inflection.

over ...

6 cont

Explicitly $y = f(\frac{4}{3}) + f'(\frac{4}{3})(x - \frac{4}{3}) = L(x)$ say

is the eqn of the tangent at $x = 4/3$. Expand about $4/3$:

$$f(x) = \underbrace{f(\tfrac{4}{3}) + f'(\tfrac{4}{3})(x - 4/3)}_{L(x)} + \underbrace{\frac{f''(\frac{4}{3})}{2}(x - 4/3)^2}_{0} + \frac{f^{(3)}(\frac{4}{3})}{6}(x - 4/3)^3 + \text{oh}$$

$$f(x) - L(x) = \frac{6}{6}(x - 4/3)^3$$

which changes sign as we go thru $x = 4/3$: $-$ve if $x < 4/3$, $+$ if $x > 4/3$.

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7 $g(x) = x^4(x-1)(x-3) = x^4(x^2 - 4x + 3)$

$= x^6 - 4x^5 + 3x^4$

$= 0 + 0\cdot x + 0\cdot x^2 + 0\cdot x^3 + 3x^4 - 4x^5 + x^6$

$\therefore$ $g'(0) = 0 = g''(0) = g'''(0)$ & $\frac{g^{(4)}(0)}{4!} = 3 > 0$ & 4 is even.

$\therefore$ $x = 0$ is a local min.

or near $x = 0$ $g(x) \approx x^4(0-1)(0-3) = 3x^4$

which has a local min at $x = 0$
