(1) $$\sum_{n=2}^{5}\left(\frac{3}{n(n-1)}+\frac{2}{4^n}\right) = 3\left(\frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+\frac{1}{4.5}\right)+2\left(\frac{1}{4^2}+\frac{1}{4^3}+\frac{1}{4^4}+\frac{1}{4^5}\right)$$

$$= \frac{6569}{2560} = 2.566..$$

$$S = \sum_{n=2}^{\infty}\left(\frac{3}{n(n-1)}+\frac{2}{4^n}\right) = 3\sum_{n=2}^{\infty}\frac{1}{n(n-1)} + 2\left(\sum_{n=0}^{\infty}\frac{1}{4^n} - \left(1+\frac{1}{4}\right)\right)$$

$$= 3S_1 + 2(S_2 - 5/4) \quad (*)$$

But $\frac{1}{n(n-1)} = \frac{1}{n-1} - \frac{1}{n}$ so $$\sum_{n=2}^{N}\frac{1}{n(n-1)} = \left(\frac{1}{1}-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\dots\left(\frac{1}{N-1}-\frac{1}{N}\right)$$
$$= 1 - \frac{1}{N}$$

$$\Rightarrow \sum_{n=2}^{\infty}\frac{1}{n(n-1)} = \lim_{N\to\infty}\left(1-\frac{1}{N}\right) = 1 = S_1 \quad \text{(telescoping sum)}$$

Hence since $S_2 = \frac{1}{1-1/4}$ (geometric series)
$= \frac{4}{3}$

$(*) \Rightarrow$ $$S = 3.1 + 2\left(\frac{4}{3}-\frac{5}{4}\right) = 19/6.$$

---

(2) Claim $$(\log n) \underset{(a)}{<} (n^{\alpha}) \underset{(b)}{<} (n) \underset{(c)}{<} (2^n) \underset{(d)}{<} (e^n) \underset{(e)}{<} (n!) \underset{(f)}{<} (n^n)$$

(a) $$\lim_{n\to\infty}\frac{\log n}{n^{\alpha}} = \lim_{m\to\infty}\frac{\log(m^{1/\alpha})}{m}$$
$$\overset{m=n^{\alpha}}{=} \lim_{m\to\infty}\left(\frac{1}{\alpha}\right)\frac{\log m}{m} = \frac{1}{\alpha}\cdot 0 = 0.$$

(b) $$\lim_{n\to\infty}\frac{n^{\alpha}}{n} = \lim_{n\to\infty}\frac{1}{n^{1-\alpha}} = 0 \quad \text{since } 1-\alpha>0$$

[Proof: given $\varepsilon>0$ choose N_ε so $\frac{1}{N_\varepsilon^{1-\alpha}}<\varepsilon \iff N_\varepsilon > \varepsilon^{\frac{1}{1-\alpha}}$

then $n \geq N_\varepsilon \Rightarrow \frac{1}{n} \leq \frac{1}{N_\varepsilon} \Rightarrow \frac{1}{n^{1-\alpha}} \leq \frac{1}{N_\varepsilon^{1-\alpha}} < \varepsilon$.]

(c) $$\lim_{n\to\infty}\frac{n}{2^n} = \lim_{n\to\infty} a_n \,;\quad \frac{a_{n+1}}{a_n} = \frac{n+1}{2^{n+1}}\cdot\frac{2^n}{n} = \left(1+\frac{1}{n}\right)\cdot\frac{1}{2} \leq 1 \Rightarrow (a_n) \text{ is}$$
decreasing & bdd below by 0 so $\lim_{n\to\infty} a_n = L$ exists. But
$a_{n+1} = \left(1+\frac{1}{n}\right)\frac{1}{2}a_n \Rightarrow L = (1+0)\frac{1}{2}L \Rightarrow L = 0$. Hence
$\lim_{n\to\infty}\frac{n}{2^n} = 0$.

(d) $e>2 \Rightarrow \left(\frac{2}{e}\right)<1 \Rightarrow \lim_{n\to\infty}\left(\frac{2}{e}\right)^n = 0$

(e) $\lim_{n\to\infty} \frac{e^n}{n!} = 0$ as n^{th} term of the conv. exponential series.

(f) $\lim_{n\to\infty} \frac{n!}{n^n}$; $a_n = \frac{n!}{n^n} \Rightarrow \frac{a_{n+1}}{a_n} = \frac{(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{n!}$

$$= \frac{(n+1)\ n^n}{(n+1)^n (n+1) \cdot 1}$$

$$= \frac{1}{(1+\frac{1}{n})^n} \to \frac{1}{e} < 1$$

Hence (a_n) is decreasing and pos. Thus

$\lim_{n\to\infty} a_n \ \exists = L$ say so $a_{n+1} = \frac{1}{(1+\frac{1}{n})^n} a_n \Rightarrow L = \frac{1}{e} L$

$\Rightarrow L = 0.$

Question can you find a sequence going to infinity slower than (a_n) all $a_n = \underbrace{\log \log \log \ldots (n)}_{\text{any finite number of logs}}$?

or faster than all (b_n) : $b_n = \left((n^{n})^{n^{\cdot^{\cdot^{\cdot^{n}}}}} \right)$

③ (a) $a_n = \frac{2}{n+1}$ $\quad b_n = \frac{1}{n}$: Since. $a_n \geq b_n$

$\Leftrightarrow \frac{2}{n+1} \geq \frac{1}{n}$

$\Leftrightarrow 2n \geq n+1 \Leftrightarrow n \geq 1$

& $\sum_1^\infty b_n)\ \mathcal{D}$ so does $\sum_1^\infty a_n)$ by comparison

OR: Since (a_n) is pos. and decreasing & $n a_n = \frac{2n}{n+1} \to 2 \neq 0 \Rightarrow \mathcal{D}$ by Abel's test.

(b) $a_n = \frac{(\log n)^n}{n^n} \Rightarrow a_n^{1/n} = \frac{\log n}{n} \to 0 < 1$ so $\sum_1^\infty a_n\ \mathcal{C}$ by Cauchy.

(c) $a_n = \frac{n!}{e^n}$ $\quad \frac{a_{n+1}}{a_n} = \frac{(n+1)!}{e^{n+1}} \cdot \frac{e^n}{n!} = \frac{n+1}{e} \geq \frac{3}{e} > 1 \ \forall n \geq 2$

so by D'Alembert's (Ratio) test I $\sum_1^\infty a_n\ \mathcal{D}$.

(d) $a_n = \frac{3^n (n!)^2}{(2n)!} \Rightarrow \frac{a_{n+1}}{a_n} = \frac{3^{n+1} (n+1)!\,(n+1)!\,(2n)!}{(2(n+1))!\ 3^n\ n!\,n!}$

$$= \frac{3\,(n+1)(n+1)}{(2n+2)(2n+1)} = \frac{3}{2}\frac{(n+1)}{(2n+1)} = \frac{3}{2}\left(\frac{1+\frac{1}{n}}{2+\frac{1}{n}}\right)$$

$\to \frac{3}{4} < 1.$

Hence by D'Alembert's (Limit Ratio) test II $\sum_{n=1}^\infty a_n\ \mathcal{C}$.

④ Just notice that $a_n \geq 0 \Rightarrow S_{n+1} = S_n + a_{n+1} \geq S_n$ so (S_n) is increasing. Hence $S_n \to S \in \mathbb{R} \Leftrightarrow (S_n)$ is bounded above.

(where $S_n = a_1 + a_2 + \ldots + a_n \ \forall n \in \mathbb{N}$)