(1) $$\sum_{n=1}^{\infty}\left(\frac{1}{n(n+1)}+\frac{2}{5^n}\right) = \sum_{n=1}^{\infty} a_n$$

$$s_1 = \frac{1}{2}+\frac{2}{5} = \frac{9}{10}$$

$$s_2 = s_1 + \frac{1}{6}+\frac{2}{25} = \frac{86}{75}$$

$$s_3 = s_2 + \frac{1}{12}+\frac{2}{125} = 1.246$$

$$s_n = \sum_{j=1}^{n}\frac{1}{j(j+1)} + 2\sum_{j=1}^{n}\frac{1}{5^j}$$

$= r_n + 2t_n$ where

$$r_n = \sum_{j=1}^{n}\left(\frac{1}{j}-\frac{1}{j+1}\right)$$ and $$t_n = \frac{1}{5}+\cdots\frac{1}{5^n} = \frac{1}{5}\left(1+\cdots+\frac{1}{5^{n-1}}\right)$$

$$= \left(\frac{1}{1}-\cancel{\frac{1}{2}}\right)+\left(\cancel{\frac{1}{2}}-\cancel{\frac{1}{3}}\right)+\left(\cancel{\frac{1}{3}}-\cancel{\frac{1}{4}}\right)+\cdots\left(\cancel{\frac{1}{n}}-\frac{1}{n+1}\right) \qquad = \frac{1}{5}\,\frac{1-\frac{1}{5^n}}{1-\frac{1}{5}} = \frac{1-\frac{1}{5^n}}{5-1}$$

$$= 1-\frac{1}{n+1} \qquad = \frac{1-\frac{1}{5^n}}{4}$$

$$\therefore\ s_n = 1-\frac{1}{n+1}+\frac{2}{4}\left(1-\frac{1}{5^n}\right)$$

$$\therefore\ \sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} s_n = 1-0+\frac{2}{4}(1-0) = 3/2$$

where limit theorems have been used in the penultimate step.

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(2) $$a_n = \frac{4n^2-3n+2}{n^2+n} = \frac{4-\frac{3}{n}+\frac{2}{n^2}}{1+\frac{1}{n}} \quad (\div\, n^2)$$

so $$\lim_{n\to\infty} a_n = \frac{4-0+0}{1+0} = 4 \neq 0$$

$$\therefore\ \sum_{n=1}^{\infty} a_n\ \mathcal{D}.$$

3 $$a_n = \frac{5^n (n!)^2}{(2n)!}$$

so $$\frac{a_{n+1}}{a_n} = \frac{5^{n+1}\,(n+1)!\,(n+1)!\,(2n)!}{(2(n+1))!\;5^n\;n!\,n!}$$

$$= \frac{5\,(n+1)!\,(n+1)!\,\cancel{(2n)!}}{n!\;n!\;(2n+2)(2n+1)\cancel{(2n)!}}$$

$$= \frac{5\,(n+1)(n+1)}{(2n+2)(2n+1)} = \frac{5\left(1+\frac{1}{n}\right)\left(1+\frac{1}{n}\right)}{\left(2+\frac{2}{n}\right)\left(2+\frac{1}{n}\right)}$$

so $$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \frac{5(1+0)(1+0)}{(2+0)(2+0)} = \frac{5}{4} > 1.$$

$\therefore \sum_{n=1}^{\infty} a_n$ $\mathcal{D}$ (to ∞) by D'Alembert's ratio test.

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4 $$0 < a_n = \frac{10n+1}{n(n+1)(n+2)} < \frac{10n+1}{n.n.n} < \frac{10n+n}{n^3} = \frac{11n}{n^3} = \frac{11}{n^2}$$

so $$0 < a_n < \frac{11}{n^2}$$

Now $$\int_1^T \frac{11}{x^2}\,dx = 11\left(-\frac{1}{x}\right)\Big|_1^T = 11\left(-\frac{1}{T} - \left(-\frac{1}{1}\right)\right)$$

$$= 11\left(1 - \frac{1}{T}\right) \to 11 \text{ as } T \to \infty.$$

$\therefore$ by the integral test, $\sum_{n=1}^{\infty} \frac{11}{n^2}$ $\mathcal{C}$.

$\therefore$ by the comparison test, $\sum_{n=1}^{\infty} a_n$ $\mathcal{C}$ also. //