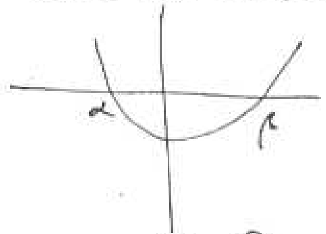
Elements of Analysis math252-08B 2008 Ass 1 Solutions

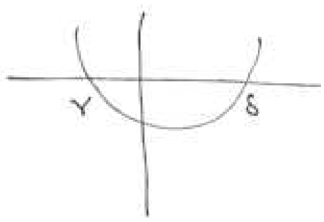
(1) $\{x : |x^2+x-2| < 1\} = S \subset \mathbb{R}$. Then $|x^2+x-2| < 1$

$\iff \quad -1 < x^2+x-2 < 1$

$\iff \quad 0 < x^2+x-1$ and $x^2+x-3 < 0$.

$\qquad 0 < (x-\alpha)(x-\beta)$ and $(x-\gamma)(x-\delta) < 0$





Using the quadratic formula

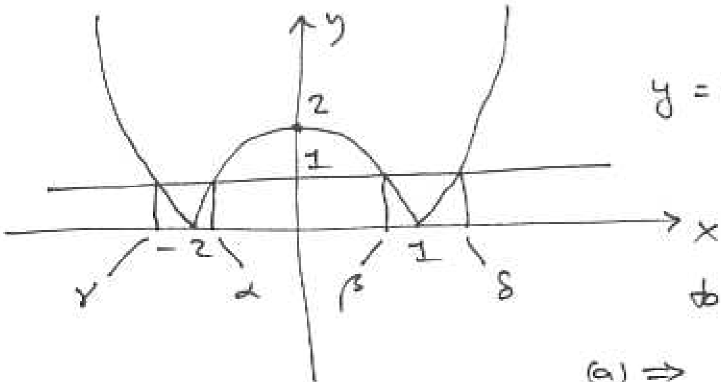
$\alpha = \frac{-1-\sqrt{5}}{2}$, $\beta = \frac{-1+\sqrt{5}}{2}$, $\gamma = \frac{-1-\sqrt{12}}{2}$ $\delta = \frac{-1+\sqrt{12}}{2}$

and $\quad \gamma < \alpha < \beta < \delta$

so $\quad S = ((-\infty,\alpha) \cup (\beta,\infty)) \cap (\gamma,\delta)$

$\qquad = (\gamma,\alpha) \cup (\beta,\delta) = \left(\frac{-1-\sqrt{12}}{2}, \frac{-1-\sqrt{5}}{2}\right) \cup \left(\frac{-1+\sqrt{5}}{2}, \frac{-1+\sqrt{15}}{2}\right)$

OR



$y = |x^2+x-2| = |x+2||x-1|$

so the region S corresponds

to the x such that $y < 1$

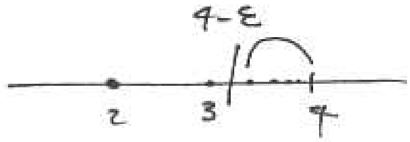
(2) Let $\beta = \text{glb}(S)$. Since $\overset{(a)\Rightarrow}{}$ b is a l.b. for S, $b \le \beta$

If $b < \beta$ (??) let $\varepsilon = \beta - b > 0$. By (b) $\exists\, x \in S$ with

$b \le x < b + \varepsilon = b + \beta - b = \beta \Rightarrow x < \beta$, a contradiction (!!)

since β is a l.b. Hence $b = \beta$ and $b = \text{glb}(S)$.

$$S = \left\{ 4 - \tfrac{2}{n},\ n = 1, 2, 3 \ldots \right\}$$
$$= \left\{ 2,\ 3,\ 3\tfrac{1}{3},\ 3\tfrac{1}{2},\ 3\tfrac{3}{5}, \ldots \right\}$$



Then (a) $2 = glb(S)$ since $2 \leq 4 - \frac{2}{n}$

$\iff \frac{2}{n} \leq 2$

$\iff 1 \leq n$ which is true, so 2 a lb.

and $2 \in S$, so it must be the glb(S).

Since $4 - \frac{2}{n} < 4 \quad \forall n \in \mathbb{N}$, 4 is an u.b.

Given $\varepsilon > 0$ we want n so $4 - \varepsilon < 4 - \frac{2}{n} \leq 4$

$\iff \frac{2}{n} < \varepsilon \iff \frac{2}{\varepsilon} < n$

and we can always find such an n. Hence, by the theorem, $4 = lub\, S$.

(b) Let $a_n = \dfrac{3n+1}{n+4}$ and let $\varepsilon > 0$ be given. Working back, we want N_ε so $|a_n - 3| < \varepsilon$

$$\iff \left| \frac{3n+1}{n+4} - 3 \right| < \varepsilon$$

$$\iff \left| \frac{3n+1-3n-12}{n+4} \right| < \varepsilon \iff \left| \frac{-11}{n+4} \right| < \varepsilon$$

$$\iff \frac{11}{n+4} < \varepsilon$$

$$\iff \frac{11}{\varepsilon} < n+4 \iff n > \frac{11}{\varepsilon} - 4 \quad \boxtimes$$

Let N_ε be any natural number with $\boxed{N_\varepsilon > \frac{11}{\varepsilon} - 4.}$

Then if $n \geq N_\varepsilon$ we have $n \geq N_\varepsilon > \frac{11}{\varepsilon} - 4 \Rightarrow n > \frac{11}{\varepsilon} - 4$

Hence $\boxtimes$ is satisfied, hence $|a_n - 3| < \varepsilon$. Thus

$\forall \varepsilon > 0 \ \exists N_\varepsilon \in \mathbb{N}$ (via $\boxed{\rightarrow}$) so $n \geq N_\varepsilon \Rightarrow |a_n - 3| < \varepsilon$.

Therefore $\displaystyle\lim_{n \to \infty} \frac{3n+1}{n+4} = 3$. //