<u>Theorem</u> [Alternating Series Test] If $a_n \geq 0 \; \forall n \in \mathbb{N}$, and $a_n \to 0$ then $L = \sum_{n=1}^{\infty} (-1)^{n+1} a_n = a_1 - a_2 + a_3 - a_4 + \dots$ converges and if S_n is the n'th partial sum, $|L - S_n| \leq a_{n+1}$.

<u>Proof</u> If n is even, $n = 2m$ then

$$S_{2m+2} = S_{2m} + (a_{2m+1} - a_{2m+2}) \geq S_{2m} \quad \forall m \in \mathbb{N}.$$

Hence the even subsequence is <u>increasing</u>.

Also
$$\begin{aligned} S_{2m} &= (a_1 - a_2) + (a_3 - a_4) + \dots \; (a_{2m-1} - a_{2m}) \\ &= a_1 - (a_2 - a_3) - (a_4 - a_5) \dots - (a_{2m-2} - a_{2m-1}) - a_{2m} \\ &= a_1 - (\text{something +ve or zero}) \end{aligned}$$

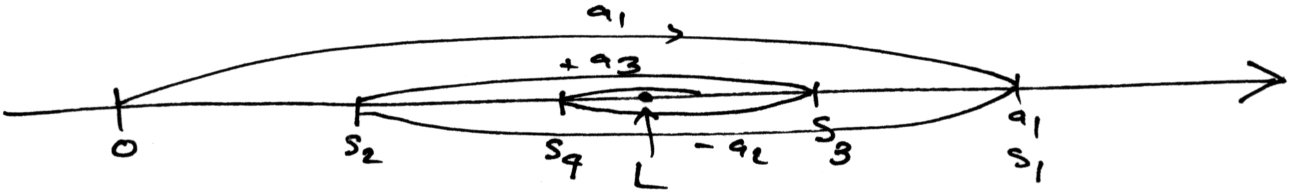
$$\Rightarrow \quad S_{2m} \leq a_1$$

so the even subsequence is bounded above. Thus $S_{2m} \to L$, for some $L \in \mathbb{R}$.

But
$$S_{2m+1} = S_{2m} + a_{2m+1} \Rightarrow$$
$$\lim_{m \to \infty} S_{2m+1} = \lim_{m \to \infty} S_{2m} + \lim_{m \to \infty} a_{2m+1} = L + 0 = L.$$

Hence, the odd subsequence tends to L also. Therefore (exercise)

$$\lim_{n \to \infty} S_n = L \quad \text{so} \quad \sum_{n=1}^{\infty} (-1)^{n+1} a_n \text{ converges.}$$



Now a similar calculation shows the odd sequences decreases with therefore $S_{2m} \leq L \leq S_{2m+1} \quad \forall m \in \mathbb{N}$.

Hence $|L - S_{2m}| \leq S_{2m+1} - S_{2m} = a_{2m+1}$ and

$$|S_{2m+1} - L| = S_{2m+1} - L \leq S_{2m+1} - S_{2m+2} = a_{2m+2}$$

Thus $|S_n - L| \leq a_{n+1} \quad \forall n \in \mathbb{N}$. //