Paper Code: MATH252-09B(HAM)

THE UNIVERSITY OF
WAIKATO
Te Whare Wānanga o Waikato

2009 B SEMESTER EXAMINATIONS

DEPARTMENT:	Mathematics
PAPER TITLE:	Elements of Analysis
TIME ALLOWED:	Two Hours
NUMBER OF QUESTIONS IN PAPER:	Six
NUMBER OF QUESTIONS TO BE ANSWERED:	Four
VALUE OF EACH QUESTION:	All questions are of equal value.
GENERAL INSTRUCTIONS:	Answer **Four** questions only.
SPECIAL INSTRUCTIONS:	Nil
CALCULATORS PERMITTED:	YES.

1. (a) Let $\varepsilon > 0$ and $x \in \mathbb{R}$. Show that if $|x| < \frac{\varepsilon}{2}$ then $-\varepsilon < 2x < \varepsilon$.

(b) Note whether each of the following sequences converges or diverges. In the case of convergence, estimate the value of the limit.

(i) $a_n = \frac{n^2}{n^3+1} + 2^{\frac{1}{n}} + \frac{\log(n)}{n}, n \in \mathbb{N}$.

(ii) $a_n = \frac{1+2^n}{2^n}$, $n \in \mathbb{N}$.

(c) Let (a_n) and (b_n) be convergent sequences with limit values L and $M \in \mathbb{R}$ respectively.

Prove that $\lim_{n\to\infty} (a_n + b_n) = L + M$.

(d) Given $\varepsilon > 0$, find $N_\varepsilon \in \mathbb{N}$ to prove that if $a_n = \frac{4n+1}{n+1}$ then $\lim_{n\to\infty} a_n = 4$.

2. (a) Let (a_n) be a sequence. What do we mean when we say the series

$$\sum_{n=1}^{\infty} a_n$$

converges to a real number S?

(b) Note, giving a reason or reasons in each case, whether each of the following series converges or diverges.

(i) $\sum_{n=1}^{\infty} \left(\frac{(-1)^{n+1}}{\sqrt{n}} + \frac{1}{2n^2} \right)$,

(ii) $\sum_{n=1}^{\infty} \left(\frac{n}{n^2+1} \right)$.

(c) If $a_n > 0$ for $n \in \mathbb{N}$ and, for some α with $0 < \alpha < 1$, and for all $n \in \mathbb{N}$, $\frac{a_{n+1}}{a_n} \le \alpha$,

prove that $\sum_{n=1}^{\infty} a_n$ converges.

Question 2(d) continued on next page.

(d) Let $\log(x)$ denote the natural logarithm of x. Show that the series $\sum_{n=2}^{\infty} \frac{1}{n\log(n)}$ diverges.

$\left[\text{Hint: } \int \frac{dx}{x\log(x)} = \log(\log(x)) + C.\right]$

3. (a) Define, using ε and Δ_ε, the limit expression

$$\lim_{x\to\infty} f(x) = L.$$

Illustrate this limit by sketching the graph of $f(x) = 2 + \frac{1}{x-1}$ for $x > 1$.

(b) Note whether each of the following limits exists as a real number or as $\pm\infty$. Where it exists, estimate its value.

(i) $\lim_{x\to 2-} \frac{x^2-4}{(x-2)(x+3)}$,

(ii) $\lim_{x\to\infty} \left(2 + \sin(x) + \frac{1}{x}\right)$.

(c) Prove, using ε and δ_ε, that if $\lim_{x\to x_0} f(x) = L$ and $c > 0$ then $\lim_{x\to x_0} c.f(x) = c.L$.

(d) Define $f:[0,2] \to \mathbb{R}$ by

$$f(x) = \begin{cases} 1-x & \text{for } 0 \le x \le 1, \\ x^2 - 3x + 2 & \text{for } 1 < x \le 2. \end{cases}$$

Prove that f is differentiable at $x = 1$.

4. (a) For each $n \in \mathbb{N}$ and $x \in \mathbb{R}$ let

$$f_n(x) = \frac{x^2}{n} + 2x.$$

Show that the sequence of functions $(f_n : n \in \mathbb{N})$ converges pointwise to the function $f(x) = 2x$ on $\mathbb{R}$.

(b) Show the convergence of the sequence of functions in (a) is uniform on [0, 1] and then that the convergence is uniform on $[-c, c]$ for every real $c > 0$.

(c) Now verify each f_n in (a) is continuous and deduce $f(x)$ is continuous on $\mathbb{R}$.

(d) Let $f_n(x) = \frac{2nx}{1+n^2x^4}$, $n \in \mathbb{N}$, $0 \le x \le 1$. Show that $f_n(x) \to h_0(x) = 0$ pointwise on $[0,1]$. Show $\lim_{n\to\infty} \int_0^1 f_n \ne \int_0^1 h_0$. Deduce the convergence is not uniform.

TURN OVER

5. (a) Let $f:[a,b]\to\mathbb{R}$ be a bounded function. Show that if $\forall\varepsilon>0$ there exists a partition P_ε of $[a,b]$ such that $U(f,P_\varepsilon)-L(f,P_\varepsilon)<\varepsilon$, then f is Riemann integrable on $[a, b]$.

(b) Let $a<c<b$. If $c>0$ and $f:[a,b]\to\mathbb{R}$ is Riemann integrable on $[a, c]$ and $[c, b]$, show that f is Riemann integrable on $[a, b]$.

(c) Let $f(x)$ be Riemann integrable on $[0, 2]$. Define $F(x)=\int_0^x f(t)dt$.

Prove that F is uniformly continuous on $[0, 2]$.

(d) Illustrate your result in (c) when

$$f(x)=\begin{cases} x & x\le 1,\\ 2 & x>1,\end{cases}$$

by finding $F(x)$ as an explicit function of x and proving it is continuous at $x=1$.

6. (a) Let $f(x)=\sum_{n=0}^{\infty}a_nx^n$ be a power series. Explain what it means to say "the real number R_f which satisfies $0<R_f<\infty$ is the radius of convergence of $f(x)$".

(b) With the notation in (a), prove that if $\lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|$ exists, its value is that of the radius of convergence of $f(x)$. Use this to find the radius of convergence of $f(x)=\log(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\ldots+\frac{(-1)^{n+1}}{n}x^n+\ldots$ with centre $x=0$.

(c) Give the general form of the Taylor expansion for a function $f(x)$ about $a=0$ with 2 terms and a remainder of order 2 in the form $f(0+h)=f(0)+\boxed{}+\boxed{}$.

(d) Now let $f(x)=x^2(x+3)$ and for given h, find the corresponding value of θ in the remainder term. Then find the nature of the critical point of $y=f(x)$ at $x=0$ and the coordinates of the point of inflection.