Paper Code: MATH252-08B(HAM)

THE UNIVERSITY OF WAIKATO
Te Whare Wānanga o Waikato

2008 B SEMESTER EXAMINATIONS

DEPARTMENT:	Mathematics
PAPER TITLE:	Elements of Analysis
TIME ALLOWED:	Two Hours
NUMBER OF QUESTIONS IN PAPER:	Six
NUMBER OF QUESTIONS TO BE ANSWERED:	Four
VALUE OF EACH QUESTION:	All questions are of equal value
GENERAL INSTRUCTIONS:	Answer **Four** questions only.
SPECIAL INSTRUCTIONS:	Nil
CALCULATORS PERMITTED:	YES.

TURN OVER

1. (a) Define, using ε and N_ε, the expression

$$\lim_{n\to\infty} a_n = L.$$

(b) Note whether each of the following sequences converge or diverge. In the case of convergence, estimate the value of the limit.

(i) $a_n = \dfrac{n+1}{n-1} + \dfrac{2\log n}{n}, n \in \mathbb{N}.$

(ii) $a_n = \dfrac{1}{2^n} + 2^n,\ n \in \mathbb{N}.$

(c) Let (a_n) be a convergent sequence with limit value $L \in \mathbb{R}$, and let $c > 0$.
Prove that $\lim_{n\to\infty} c.a_n = c.L$.

(d) Given $\varepsilon > 0$, find $N_\varepsilon \in \mathbb{N}$ to prove that if $a_n = \dfrac{2n}{n+1}$ then $\lim_{n\to\infty} a_n = 2.$

2. (a) Let (a_n) be a sequence. Give the definition of

$$\sum_{n=1}^{\infty} a_n .$$

(b) Note, giving a reason in each case, whether each of the following series converges or diverges. In the case of convergence, estimate the sum of the series by summing the first four terms.

(i) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^3+1},$

(ii) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{n^2} - \frac{1}{(n+1)^2} \right).$

(c) If $a_n \geq 0$ for all $n \in \mathbb{N}$ and, for some $\delta > 1$, $\lim_{n\to\infty} n^\delta a_n = 0$, prove that $\displaystyle\sum_{n=1}^{\infty} a_n$ converges.

(d) If the series of positive terms $\displaystyle\sum_{n=1}^{\infty} a_n$ converges, show that $\displaystyle\sum_{n=1}^{\infty} {a_n}^2$ converges also.

3. (a) Define, using ε and δ_ε, the limit expression

$$\lim_{x\to x_0} f(x) = L.$$

(b) Note whether each of the following limits exists as a real number or as $\pm\infty$. Where it exists, estimate its value.

(i) $\lim_{x\to 2-} \frac{x^2-1}{(x-1)(x+2)}$,

(ii) $\lim_{x\to\infty}\left(e^{-x} + \arctan(x) + \frac{1}{x^2+1}\right)$.

(c) Prove, using ε and δ_ε, that if $\lim_{x\to x_0} f(x) = L$ and $\lim_{x\to x_0} g(x) = M$ then $\lim_{x\to x_0} f(x) + g(x) = L + M$.

(d) Define $f : [0, 2] \to \mathbb{R}$ by

$$f(x) = \begin{cases} 1 + x & \text{for} \quad 0 \le x \le 1, \\ 3x - x^2 & \text{for} \quad 1 < x \le 2. \end{cases}$$

Prove that f is differentiable at $x = 1$.

4. (a) For each $n \in \mathbb{N}$ and $x \in \mathbb{R}$ let

$$f_n(x) = \frac{x^2}{n} + x.$$

Show that the sequence of functions $(f_n : n \in \mathbb{N})$ converges pointwise to the function $f(x) = x$ on $\mathbb{R}$.

(b) Show the convergence of the sequence of functions in (a) is uniform on [0, 1] and then that the convergence is uniform on $[-c, c]$ for every real $c > 0$.

(c) Now verify each f_n is continuous and deduce $f(x)$ is continuous on $\mathbb{R}$.

(d) Let $g_n : [a, b] \to \mathbb{R}$, for $n \in \mathbb{N}$ be a sequence of continuous functions and let $g_n \to g$ uniformly on $[a, b]$. Prove that g is continuous at every point $c \in [a, b]$.

5. (a) Let $f:[a,b]\to\mathbb{R}$ be a bounded Riemann integrable function. Show that $\forall\varepsilon>0$ there exists a partition P_ε of $[a,b]$ such that $U(f,P_\varepsilon)-L(f,P_\varepsilon)<\varepsilon$.

(b) If $c>0$ and $f:[a,b]\to\mathbb{R}$ is Riemann integrable, show that $c.f$ is Riemann integrable also.

(c) Prove that if $f:[a,b]\to\mathbb{R}$ is continuous on $[a, b]$ then it is uniformly continuous on $[a, b]$. (Hint: use the theorem that every bounded sequence in $\mathbb{R}$ has a convergent subsequence.)

(d) Let $f:[0,2]\to\mathbb{R}$ be continuous. Prove that it is Riemann integrable on $[a, b]$.

6. (a) Let $f(x)=\sum_{n=0}^{\infty}a_nx^n$ be a power series. Explain what it means to say "the real number R_f which satisfies $0<R_f<\infty$ is the radius of convergence of $f(x)$".

(b) With the notation in (a), prove that if $\lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|$ exists, its value is that of the radius of convergence of $f(x)$.

(c) Using the power series $\exp(x)=1+x+\frac{x^2}{2!}+\ldots+\frac{x^n}{n!}+\ldots$ show that $R_{\exp}=\infty$ and use this to explain why $\exp'(x)=\exp(x)$ on $\mathbb{R}$.

(d) State the binomial theorem for the expansion of

$$f(x)=(1+x)^m$$

for $m\in\mathbb{R}\setminus\{0,1,2,\ldots\}$. Simplify the n'*th* term in case $m=\frac{1}{2}$ and use the expansion as far as the term in x^2 to check the value of $\sqrt{1+\frac{9}{16}}$.

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