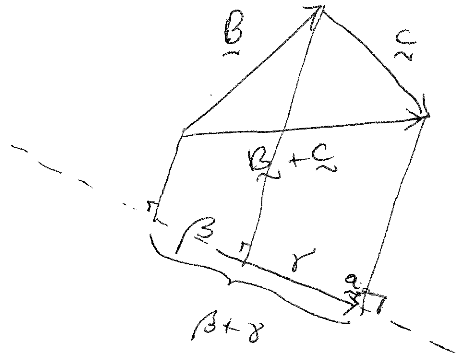
let a be a unit vector in the direction A. i.e. $a\|A\| = A$. (p1)

(i)



where A, B, C are any non zero vectors in $\mathbb{R}^3$.

Project B, C and $B+C$ onto a line through a to get lengths $\beta, \gamma, \beta+\gamma$ respectively.

Then
$$\left.\begin{aligned} \beta &= B\cdot a \\ \gamma &= C\cdot a \\ (\beta+\gamma) &= (B+C)\cdot A \end{aligned}\right\} \Rightarrow \begin{aligned} \beta+\gamma &= (\beta+\gamma) \\ B\cdot a + C\cdot a &= (B+C)\cdot a \end{aligned}$$

$$\Rightarrow \quad B\cdot a\,\|A\| + C\cdot a\,\|A\| = (B+C)\cdot a\,\|A\|$$

$$\Rightarrow \quad B\cdot A + C\cdot A = (B+C)\cdot A$$

& by commutativity of $\cdot$, $\quad A\cdot(B+C) = A\cdot B + A\cdot C.$

Then $\quad \underline{i}\cdot\underline{i} = \underline{j}\cdot\underline{j} = \underline{k}\cdot\underline{k} = 1 \quad \& \quad \underline{i}\cdot\underline{j} = \underline{i}\cdot\underline{k} = \underline{k}\cdot\underline{j} = 0 \Rightarrow$

$$A\cdot B = (a_1 i + a_2 j + a_3 k)\cdot(b_1 i + b_2 j + b_3 k)$$
$$= a_1 b_1 + a_2 b_2 + a_3 b_3.$$

So on $\mathbb{R}^n$ or $\mathbb{C}^n$ define

$$\underline{A}\cdot\underline{B} := \sum_{j=1}^{n} a_j b_j = a_1 b_1 + \dots + a_n b_n$$

---

and on $\mathbb{C}^n$ $\quad \langle A, B\rangle := A\cdot\overline{B} \quad$ (correct form of Dr Murray's def^n)

---

(2) (Thm 27) V has a basis $\{\underline{b}_1, \ldots \underline{b}_d\}$.

so. $\underline{x} \in V \Rightarrow x = x_1 \underline{b}_1 + \ldots + x_d \underline{b}_d$ unique.

dy $T: V \longrightarrow \mathbb{C}^d$ via.

$x \longrightarrow (x_1, \ldots x_d) \in \mathbb{C}^d$.

lin, 1-1 & onto. so its invertible.

---

(3) $\langle u, v\rangle = T(\underline{u}) \cdot \overline{T(v)}$

Defn $\underline{a} \cdot \underline{b} = \sum_{i=1}^{n} a_i b_i$ so. $\underline{a} \cdot \underline{b} = \underline{b} \cdot \underline{a}$ & $\underline{a} \cdot (\underline{b} + \underline{c}) = \underline{a} \cdot \underline{b} + \underline{a} \cdot \underline{c}$

so. $\overline{\langle v, u\rangle} = \overline{T(v) \cdot \overline{T(u)}} = \overline{(v_1, \ldots v_d) \cdot \overline{(u_1, \ldots, u_d)}}$

$= \overline{(v_1, \ldots v_d) \cdot (\overline{u}_1, \ldots \overline{u}_d)}$

$= \overline{v_1 \overline{u}_1 + \ldots v_d \overline{u}_d}$

$= \overline{v}_1 \overline{\overline{u}}_1 + \ldots \overline{v}_d \overline{\overline{u}}_d$

$= u_1 \overline{v}_1 + \ldots u_d \overline{v}_d = \langle u, v\rangle$ IP1

---

(4) $\langle u + u', v\rangle = T(u + u') \cdot \overline{T(v)}$

$= (T(u) + T(u')) \cdot \overline{T(v)}$.

$= T(u) \cdot \overline{T(v)} + T(u') \cdot \overline{T(v)}$

$= \langle u, v\rangle + \langle u', v\rangle$ i.e. IP2 p3

(5) $\langle x,x\rangle = 0 \iff T(x)\cdot\overline{T(x)} = 0 \iff x_1\bar{x}_1 + \dots + x_d\bar{x}_d = 0$

But $z\bar{z} = |z|^2$ $\qquad |x_1|^2 + \dots |x_d|^2 = 0 \iff x_i = 0\ \forall i$

$\quad\ = (x+iy)(x-iy) = x^2+y^2 \qquad \iff \underline{x=0}$.

---

(6) Cauchy-Schwarz.

$Re\langle u,v\rangle \le \|u\|\cdot\|v\|$

$\langle u,v\rangle \in \mathbb{C}$ so.

$\langle u,v\rangle = |\langle u,v\rangle| e^{i\theta}$ polar form.

$0 \le \theta < 2\pi$

$\Rightarrow \langle e^{-i\theta}u, v\rangle = |\langle u,v\rangle|$.

& $\Rightarrow Re\langle e^{-i\theta}u, v\rangle \le \|e^{-i\theta}u\|\cdot\|v\|$

$\quad = |\langle u,v\rangle|$

& $\|\alpha u\| = |\alpha|\,\|u\|$

Proof. $\|\alpha u\|^2 = \langle \alpha u, \alpha u\rangle \overset{IP3}{=} \alpha\langle u, \alpha u\rangle \overset{IP1}{=} \alpha\overline{\langle \alpha u, u\rangle}$

$\overset{IP3}{=} \alpha\left(\overline{\alpha\langle u,u\rangle}\right)$

$= \alpha\bar{\alpha}\overline{\langle u,u\rangle} = |\alpha|^2\langle u,u\rangle = |\alpha|^2\|u\|^2$

$\therefore$ $\|\alpha u\| = |\alpha|\,\|u\|$

$\therefore$ $|\langle u,v\rangle| \le |e^{-i\theta}|\,\|u\|\cdot\|v\| = \|u\|\cdot\|v\|$.

$|e^{-i\theta}|$ is $=1$

$= \sqrt{\cos^2(\theta)+\sin^2(\theta)} = \sqrt{1} = 1$.

---

(7) Equality in Cauchy Schwarz (in $\mathbb{R}$). Let $|\langle u,v\rangle| = \|u\|\cdot\|v\|$.

assume. $u,v \ne 0$ $\Rightarrow \left|\left\langle \frac{u}{\|u\|}, \frac{v}{\|v\|}\right\rangle\right| = 1$. $\qquad \langle \hat{w}, \hat{v}\rangle = 1$.

write $|\langle \hat{u}, \hat{v}\rangle| = 1$ $\qquad \hat{u}$ & $\hat{v}$ have norm 1.

then. $\|\hat{u}-\hat{v}\|^2 \overset{\text{Defn norm}}{=} \langle \hat{u}-\hat{v}, \hat{u}-\hat{v}\rangle = \langle \hat{u},\hat{u}\rangle - \langle \hat{w},\hat{u}\rangle - \langle \hat{u},\hat{w}\rangle + \langle \hat{w},\hat{w}\rangle$ $\qquad \hat{w} = e^{-i\theta}\hat{u}$

$= 1 - 1 - 1 + 1 = 0$

$\therefore$ By IP4 $\hat{u} - \hat{w} = 0 \Rightarrow \hat{u} = \hat{w} \Rightarrow \exists \lambda \ne 0, u = \lambda v$

$\Rightarrow \underline{u} = \left(\frac{\|u\|}{\|v\|}e^{-i\theta}\right)\underline{v} = \lambda\underline{v}$.

Theorem. 3.5. (i) $W \cap W^{\perp} = \{0\}$

(ii) W finite dimensional $\Rightarrow$ $W = (W^{\perp})^{\perp}$.

Proof (i) Let $x \in W \cap W^{\perp} \Rightarrow x \in W \Rightarrow \langle x, v \rangle = 0 \ \forall v \in W^{\perp}$

but $x \in W^{\perp}$ so we can choose $v = x \Rightarrow \langle x, x \rangle = 0 \Leftrightarrow x = 0$.

$\therefore$ $W \cap W^{\perp} = \{0\}$.

(ii). Let $w \in W \Rightarrow \langle w, v \rangle = 0 \ \forall v \in W^{\perp}$

$\forall$ $v \in W^{\perp} \Rightarrow \langle v, w \rangle = 0$ $w \in W \Rightarrow w \in (W^{\perp})^{\perp}$

$\Rightarrow$ $W \subset (W^{\perp})^{\perp}$

Claim $(W^{\perp})^{\perp} \subset W$:

Let $v \in (W^{\perp})^{\perp}$ and $w \in W$ be the best approx to v

$\Rightarrow \forall x \in W$, $x \perp (v - w)$ $\boxtimes$ (Thm 3.3).

But $w \in W \subset (W^{\perp})^{\perp}$ + $v \in (W^{\perp})^{\perp}$, a subspace

$\Rightarrow$ $v - w \in (W^{\perp})^{\perp}$ + $\boxtimes$ $\Rightarrow$ $v - w \in W^{\perp}$ also.

But $(W^{\perp}) \cap (W^{\perp})^{\perp} = \{0\} \Rightarrow v - w = 0 \Rightarrow v = w$

so $v \in W$

$\therefore$ $(W^{\perp})^{\perp} \subset W$

$\therefore$ $W = (W^{\perp})^{\perp}$. //

Theorem 3.10

Let $w' = c_1 w_1 + \dots + c_k w_k \in W$

given $v = (v - \mathrm{proj}_W v) + \mathrm{proj}_W v \in W^{\perp} \oplus W$ unique

$v = (v - w') + w'$

$w' \in W$ so $v - w' = v - \mathrm{proj}_W v \Rightarrow w' = \mathrm{proj}_W v$

The idea is given $v \in V$ and $w \in W$, since $v = \underset{\hat{W}}{w} + \underset{\hat{W^{\perp}}}{u}$ is unique

if $w = \mathrm{proj}_W v$ then $u = v - w \in W^{\perp}$

and if $u \in W^{\perp}$, $w = v - u \in W$

Have. $V = W \oplus W^{\perp}$

$v = \text{proj}_W v + (v - \text{proj}_W v)$ unique

$W = \{v_1, \dots, v_k\}$ basis

$w' = c_1 v_1 + \dots + c_k v_k \Rightarrow$ $v = w' + (v - w')$ always

uniqueness $\Rightarrow$ $w' = \text{proj}_W v \Leftrightarrow v - w' \in W^{\perp}$.

If $\{u_1, \dots, u_k\}$ is an orthonormal basis for W

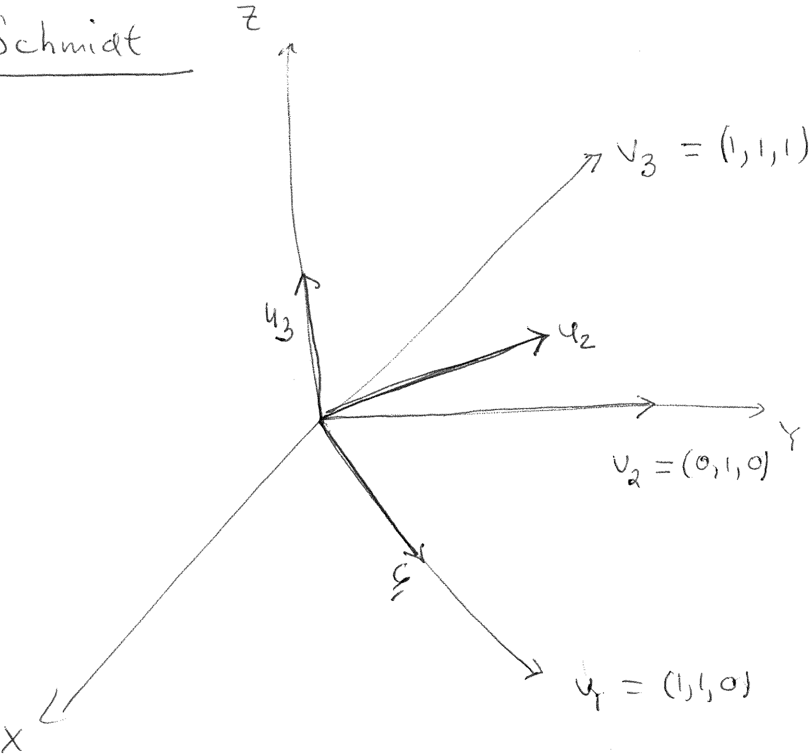
$$\langle u_i, u_j \rangle = \delta_{ij} = \begin{cases} 1 & \text{for } i = j, \\ 0 & \text{for } i \neq j. \end{cases}$$

$\therefore \text{proj}_W v = c_1 u_1 + \dots + c_k u_k. \Rightarrow$

$$\begin{pmatrix} c_1 \\ \vdots \\ c_k \end{pmatrix} = \begin{pmatrix} \langle u_i, u_j \rangle \end{pmatrix} \begin{pmatrix} c_1 \\ \vdots \\ c_k \end{pmatrix} = \begin{pmatrix} \langle v, u_1 \rangle \\ \vdots \\ \langle v, u_k \rangle \end{pmatrix} \Rightarrow \boxed{\text{proj}_W v = \langle v, u_1 \rangle u_1 + \dots + \langle v, u_k \rangle u_k}$$

Expression to compute a projection

Ex Gram-Schmidt



Direct Sum If $U \subset V$, $W \subset V$ are subspaces,

$U + W = \{u + w : u \in U, w \in W\}$ is the sum subspace.

It is a direct sum $\Leftrightarrow$ $U \cap W = \{0\}$ & then we write $U \oplus W = U + W$

and the expression, given $v \in U \oplus W$, $v = u + w$, is unique.

Gram-Schmidt process

1. $v_1 = (1,1,0) \rightarrow \quad w_1 = v_1, \quad u_1 = \frac{w_1}{\|w_1\|} \quad \& \quad W_1 = \{u_1\}$.

2. project. $\mathrm{proj}_{W_1} v_2$. $\quad w_2 = v_2 - \mathrm{proj}_{W_1} v_2 \perp W_1 \rightarrow \quad u_2 = \frac{w_2}{\|w_2\|}, \quad W_2 = \{u_1, u_2\}$

3. project. $\mathrm{proj}_{W_2} v_3 \quad w_3 = v_3 - \mathrm{proj}_{W_2} v_3 \perp W_2 \rightarrow \quad u_3 = \frac{w_3}{\|w_3\|}, \quad W_3 = \{u_1, u_2, u_3\}$

Ex: 1. $v_1 = (1,1,0)$

$$u_1 = \frac{(1,1,0)}{\|(1,1,0)\|} = \frac{1}{\sqrt{2}}(1,1,0) = \left(\tfrac{1}{\sqrt{2}}, \tfrac{1}{\sqrt{2}}, 0\right) \qquad W_1 = \left\{\left(\tfrac{1}{\sqrt{2}}, \tfrac{1}{\sqrt{2}}, 0\right)\right\}.$$

2. $v_2 = (0,1,0) \rightarrow \quad w_2 = (0,1,0) - \mathrm{proj}_{W_1}(v_2)$, $v_2 = (0,1,0)$

$$(0,1,0) - \langle v_2, u_1\rangle u_1.$$

$$(0,1,0) - \left((0,1,0)\cdot\left(\tfrac{1}{\sqrt{2}}, \tfrac{1}{\sqrt{2}}, 0\right)\right)\left(\tfrac{1}{\sqrt{2}}, \tfrac{1}{\sqrt{2}}, 0\right)$$

$$(0,1,0) - \left(\tfrac{1}{\sqrt{2}}\right)\left(\tfrac{1}{\sqrt{2}}, \tfrac{1}{\sqrt{2}}, 0\right)$$

$$\left(\tfrac{1}{2}, \tfrac{1}{2}, 0\right).$$

$$= \left(-\tfrac{1}{2}, \tfrac{1}{2}, 0\right).$$

$$\& \quad u_2 = \frac{w_2}{\|w_2\|} = \frac{\left(-\frac{1}{2}, \frac{1}{2}, 0\right)}{\sqrt{\frac{1}{4} + \frac{1}{4} + 0}} = \left(-\tfrac{1}{\sqrt{2}}, \tfrac{1}{\sqrt{2}}, 0\right).$$

$$W_2 = \{u_1, u_2\}.$$

---

3. $v_3 = (0,0,1) \longrightarrow \quad w_3 = v_3 - \mathrm{proj}_{W_2} v_3.$

$$w_3 = \underbrace{(0,0,1)}_{v_3} - \left(\langle v_3, u_1\rangle u_1 + \langle v_3, u_2\rangle u_2\right)$$

$$= (0,0,1) - \left(\left((0,0,1)\cdot\left(\tfrac{1}{\sqrt{2}}, \tfrac{1}{\sqrt{2}}, 0\right)\right)\left(\tfrac{1}{\sqrt{2}}, \tfrac{1}{\sqrt{2}}, 0\right) + \left((0\ 0\ 1)\cdot\left(-\tfrac{1}{\sqrt{2}}, \tfrac{1}{\sqrt{2}}, 0\right)\right)\left(-\tfrac{1}{\sqrt{2}}, \tfrac{1}{\sqrt{2}}, 0\right)\right)$$

$$= (0\,0\,1) - 0\left(\tfrac{1}{\sqrt{2}}, \tfrac{1}{\sqrt{2}}, 0\right) - 0\left(-\tfrac{1}{\sqrt{2}}, \tfrac{1}{\sqrt{2}}, 0\right) = (0,0,1)$$

$$\& \quad u_3 = \frac{w_3}{\|w_3\|} = (0,0,1)$$

$\therefore \{u_1, u_2, u_3\} = \left\{\left(\tfrac{1}{\sqrt{2}}, \tfrac{1}{\sqrt{2}}, 0\right), \left(-\tfrac{1}{\sqrt{2}}, \tfrac{1}{\sqrt{2}}, 0\right), (0,0,1)\right\}$ is an <u>orthonormal</u> basis for $\mathbb{R}^3$.

(a) If the dim is $n \in \mathbb{N}$ then there are n steps rather than 3.

(b) The first vector can be any vector from the basis.

(c) Its easy to program this — "the Gram-Schmidt process".

If $\{u_1, \ldots, u_j\}$ are known $w_{j+1} = v_{j+1} - \mathrm{proj}_{W_j} v_{j+1} \rightarrow u_{j+1} = \frac{w_{j+1}}{\|w_{j+1}\|}$.