Math101 - 10B Test 2 Solutions

(A) (1) $\int f(x)\,dx = \dfrac{x^2}{2} - \cos(x) + x + C$

(2) area $= \displaystyle\int_1^2 (2+3x^2)\,dx = (2x + x^3)\Big|_1^2$

$= (2.2 + 2^3) - (2.1 + 1^3)$

$= 12 - 3 = 9$

(3) $S(24) = \dfrac{24(2.24+1)(24+1)}{6}$

$= 4 \times 49 \times 25 = 49 \times 100 = 4{,}900$

$\displaystyle\lim_{n\to\infty} \frac{S(n)}{n^3} = \lim_{n\to\infty} \frac{n(2n+1)(n+1)}{6\,n^3} = \frac{1}{6}\lim_{n\to\infty}\left(\frac{2n+1}{n}\right)\left(\frac{n+1}{n}\right)$

$\displaystyle = \frac{1}{6}\lim_{n\to\infty}\left(2+\frac{1}{n}\right)\left(1+\frac{1}{n}\right) = \frac{1}{6}(2+0)(1+0) = \frac{1}{3}$

(4) $\dfrac{1}{\sqrt{2}} = \sin(x + \pi/4)$ and $-\frac{\pi}{2} \le x \le \frac{\pi}{2} \Rightarrow x = 0$

$\displaystyle \left.\frac{dx}{dy}\right|_{y=1/\sqrt{2}} = \frac{1}{\left.\frac{dy}{dx}\right|_{x=0}} = \frac{1}{\left.\cos(x+\frac{\pi}{4})\right|_{x=0}} = \frac{1}{\cos(\pi/4)} = \frac{1}{1/\sqrt{2}} = \sqrt{2}$

(5) $3\displaystyle\int (x^2+1)^2\,dx = 3\int ((x^2)^2 + 2x^2 + 1)\,dx = 3\int (x^4 + 2x^2 + 1)\,dx$

$= 3\left[\dfrac{x^5}{5} + \dfrac{2x^3}{3} + x\right] + C$

$= \frac{3}{5}x^5 + 2x^3 + 3x + C$

(6) Let $u = x^2+1 \Rightarrow$ $\displaystyle\int_0^1 \frac{2x}{x^2+1}\,dx = \int_1^2 \frac{du}{u} = \ln|u|\Big|_1^2 = \ln 2 - \ln 1$

$du = 2x\,dx$

$x = 0 \Rightarrow u = 1$ $\quad = \ln 2$

$x = 1 \Rightarrow u = 2$

(7) $\displaystyle\int\left(\frac{1}{x+1} + \frac{1}{\sqrt{1-x^2}} + e^{2x}\right)dx = \ln|x+1| + \sin^{-1}(x) + \frac{1}{2}e^{2x} + C$

(8) $u = x^2$ $\quad y = F(x) = \displaystyle\int_0^u \frac{e^t}{t}\,dt \Rightarrow \frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$

so $F'(x) = \dfrac{dy}{dx} = \dfrac{e^u}{u}.2x = \dfrac{e^{x^2}.2x}{x^2} = \dfrac{2e^{x^2}}{x}$

Part A

(9)
$$\frac{1}{(x+1)(x-2)} = \frac{A}{x+1} + \frac{B}{x-2} \Rightarrow$$
$$1 = A(x-2) + B(x+1) \Rightarrow \text{letting } x = -1$$
$$1 = A(-1-2) \Rightarrow \boxed{A = -1/3} \text{ \& letting } x = 2$$
$$1 = 0 + B(2+1) \Rightarrow \boxed{B = 1/3}$$
so $A = -1/3,\ B = 1/3$.

(10)
$$\int_1^2 x \ln(x)\,dx = \int_1^2 \underbrace{\ln x}_{u}\ \underbrace{x\,dx}_{dv}.$$

$u = \ln(x)$, $du = \frac{1}{x}dx$, $\frac{dv}{dx} = x$, $v = \frac{x^2}{2}$

$$= uv\Big|_1^2 - \int_1^2 v\,du$$
$$= \frac{x^2}{2}\ln(x)\Big|_1^2 - \int_1^2 \frac{x^2}{2}\cdot\frac{1}{x}\,dx$$
$$= \left(\frac{2^2}{2}\ln(2) - \frac{1^2}{2}\ln(1)\right) - \int_1^2 \frac{x}{2}\,dx$$
$$= 2\ln(2) - 0 - \left(\frac{x^2}{4}\Big|_1^2\right)$$
$$= \ln(4) - \left(\frac{2^2}{4} - \frac{1^2}{4}\right) = \ln(4) - 3/4 \quad //$$

Part B 1 (a) $u = f(x) > 0 \Rightarrow (\ln|f(x)|)' = \frac{d}{dx}\ln(u) = \frac{d}{du}\ln(u)\frac{du}{dx}$
$$= \frac{1}{u}\frac{du}{dx} = \frac{f'(x)}{f(x)}$$

& if. $f(x) < 0$ let $u = -f(x) \Rightarrow$
$|f(x)| = u$
$$(\ln|f(x)|)' = \frac{1}{u}\frac{du}{dx} = \frac{1}{-f(x)}(-f'(x)) = \frac{f'(x)}{f(x)} \quad //$$

Hence
$$f(x) = \frac{(x+2)(x^3+2)^{1/2}}{(x+5)^2} \Rightarrow$$
$$\ln|f(x)| = \ln|x+2| + \frac{1}{2}\ln|x^3+2| - 2\ln|x+5|$$
$$\Rightarrow \frac{f'(x)}{f(x)} = \frac{1}{x+2} + \frac{1}{2}\frac{2x^2}{x^3+2} - \frac{2}{x+5}$$
so
$$f'(0) = f(0)\left[\frac{1}{2} + 0 - \frac{2}{5}\right] = \frac{\cancel{2}\sqrt{2}}{5^2}\times\frac{3}{\cancel{10}_{5}} = \frac{3\sqrt{2}}{125} \quad //$$

(b) $f(x) = \ln|\sec(x) + \tan(x)| \Rightarrow f'(x) = \dfrac{\sec(x)\tan(x) + \sec^2 x}{\sec(x) + \tan(x)}$
$$= \frac{\sec(x)\,\cancel{(\sec(x) + \tan(x))}}{\cancel{\sec(x) + \tan(x)}} = \sec(x)$$

Hence $\int \sec(x)\,dx = \ln|\sec(x) + \tan(x)| + C$

Part B

I (c) $\int e^x \sin(2x)\,dx = Ae^x \sin(2x) + Be^x \cos(2x) + C$

$\Rightarrow A(e^x \sin(2x))' + B(e^x \cos(2x))' = e^x \sin(2x)$

$\Rightarrow A(e^x \sin(2x) + 2e^x \cos(2x)) = e^x \sin(2x)$ OR

$+ B(e^x \cos(2x) - 2e^x \sin(2x))$

$e^x \sin(2x) = e^x \sin(2x)[A - 2B] + e^x \cos(2x)[2A + B] \Rightarrow$

$2A + B = 0$ and $A - 2B = 1 \Rightarrow A - 2(-2A) = 1 \Rightarrow A = 1/5$

& $B = -2/5$

2 (a) $\int_0^1 \frac{x+1}{(x+2)(x+3)}\,dx = I$

$\frac{x+1}{(x+2)(x+3)} = \frac{A}{x+2} + \frac{B}{x+3}$

$x+1 = A(x+3) + B(x+2)$

$x = -2 \Rightarrow A = -1$
$x = -3 \Rightarrow B = 2$ } Hence $I = \int_0^1 \frac{-1}{x+2} + \frac{2}{x+3}\,dx.$

$= (2\ln|x+3| - \ln|x+2|)\Big|_0^1$

$= (2\ln 4 - \ln 3) - (2\ln(3) - \ln(2))$

$= 4\ln 2 - \ln 3 - 2\ln 3 + \ln 2$

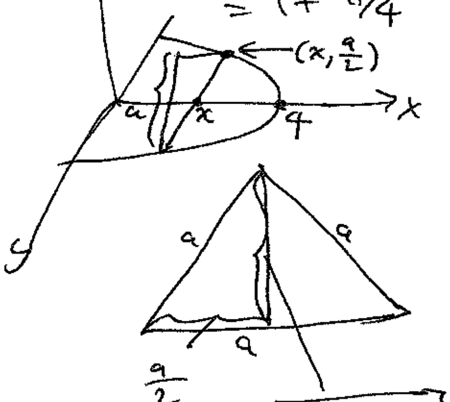
$= 5\ln 2 - 3\ln 3 = \ln\left(\frac{2^5}{3^3}\right)$

(b) $\int_0^1 \frac{x^2+2}{x^2+1}\,dx = \int_0^1 \frac{(x^2+1)+1}{x^2+1}\,dx = \int_0^1 \left(\frac{x^2+1}{x^2+1} + \frac{1}{x^2+1}\right)dx$

$= \int_0^1 \left(1 + \frac{1}{x^2+1}\right)dx = x + \tan^{-1}(x)\Big|_0^1 = (1 + \tan^{-1}(1)) - 0$

$= 1 + \pi/4$

(c)



$\sqrt{a^2 - \left(\frac{a}{2}\right)^2}$

area $= \frac{1}{2} a \times \frac{\sqrt{3}}{2} a$ $\quad = \sqrt{\frac{a^2 3}{4}}$

$= \frac{\sqrt{3}}{4} a^2$ $\quad = \frac{\sqrt{3}}{2} a$

and $x^2 + \left(\frac{a}{2}\right)^2 = 4^2 \Rightarrow \frac{a^2}{4} = 4^2 - x^2$

$\text{Vol} = 2\int_0^4 (\text{area of } \Delta)\,dx.$

$= 2\int_0^4 \sqrt{3}\left(\frac{a^2}{4}\right)dx$

$= 2\int_0^4 \sqrt{3}\,(4^2 - x^2)\,dx$

$= 2\sqrt{3}\left[4^2 x - \frac{x^3}{3}\Big|_0^4\right]$

$= 2\sqrt{3}\,\frac{2}{3}\,4^3 = \frac{4^4}{\sqrt{3}}$