Relationships between the integer conductor and k'th root functions

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The conductor of a rational integer is the product of the primes which divide it. The lower k'th root is the largest k'th power divisor, and the upper k'th root the smallest k'th power multiple. This paper examines the relationships between these arithmetic functions and their Dirichlet series. It is shown that the conductor is the limit of the upper k'th roots in two different ways as k tends to infinity. The average order of the functions is derived.

Key Words: Integer square root, integer k'th root, Dirichlet series

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1. INTRODUCTION

The integer conductor is $N(n) = \prod_{p|n} p$. For each whole number k let the integer lower k'th root be defined by

$$\rho_k(n) = \prod_{p^\alpha || n} p^{\lfloor \frac{\alpha}{k} \rfloor}$$

and the integer upper k'th root by

$$\rho^k(n) = \prod_{p^\alpha || n} p^{\lceil \frac{\alpha}{k} \rceil}$$

The properties of these two k'th root function families, and their close relationships to the conductor, are studied in this paper For example in Section 5, the sequence of Dirichlet series for the k'th root tends pointwise and in an appropriate space of Dirichlet series to the series for the conductor, as $k \to \infty$. More fundamentally $N(n) = \rho^k(n)$ for $k \geq k_n$.

In Section 2 a list of elementary properties of the functions is listed. When used in the paper they are referenced by their property number.

In Section 3 the Dirichlet series for the lower k'th root is shown to have a closed form as a rational function of zeta function values. This is not so for the upper root which has a more complicated series, each term involving a (finite) product. A closed form is also obtained including a product of zeta functions and an Euler product of order k.

In Section 4 the average order of the roots is investigated. Again for the lower roots $\rho_k(n)$ the asymptotic order is readily determined to be $\zeta(k-1)/\zeta(k)$ for $k > 2$ and $\log n/\zeta(2)$ for $k = 2$. For the upper root, the $k = 2$ case is determined directly (with average order $x\zeta(3)/2\zeta(2)$), with the cases $k > 2$ determined using results on the asymptotic order of $T_u(x) := \sum_{n \le x} N(un)$ as a function of (square free) u and the limiting value of a certain partial sum function $A_k(x)$.

The function T_u is shown to have an interesting looking property. For each square free positive integer u:

$$\sum_{d|u} \mu(d) T_u(\frac{x}{d}) = u \sum_{d|u} \mu(d) T_d(\frac{x}{d}).$$

2. PROPERTIES OF THE FUCTIONS

Below we state (without proof except for (15)) some properties of the k'th root and conductor functions:

(1) $N(a)N(b) = N(ab)N((a,b))$ where (a,b) is the gcd of a and b.

(2) The functions ρ_k and ρ^k are multiplicative for all $k \ge 1$.

(3) $N(n)|n$ and there exists an l for which $n|N(n)^l$. If $d|n$ and $n|d^l$ then $N(n)|d$.

(4) The equation $n = ab^k$ with a k-free holds if and only if $b = \rho_k(n)$.

(5) The equation $an = b^k$ with a k-free holds if and only if $b = \rho^k(n)$.

(6) If $b^k|n$ and if whenever $c^k|n$ then $c|b$ then necessarily $b = \rho_k(n)$.

(7) If $n|b^k$ and if whenever $n|c^k$ then $b|c$ then necessarily $b = \rho^k(n)$.

(8) For all $k \ge 1$ $\rho_{k+1}(n)|\rho_k(n)$, $\rho^{k+1}(n)|\rho^k(n)$, and $\rho_k(n)|\rho^k(n)$.

(9) If $n = ab^k$ and $cn = d^k$ with a and c k-free, then $ac = m^k$ where m is square free, being the product of the primes in n which do not appear to the k'th power.

(10) For all $k \ge 1$, $\rho^k(n) = N(a)\rho_k(n)$, where $n = ab^k$ and a is k-free.

(11) For all $k \ge 1$, $n^{\frac{1}{k}} \le \rho^k(n) \le n$ and $1 \le \rho_k(n) \le n^{\frac{1}{k}}$.

(12) For all n there exists a k_n such that $N(n) = \rho^k(n)$ for all $k \ge k_n$.

(13) For all n and $k \ge 1$ there exists an m such that

$$N(n) = \overbrace{\rho^k \circ \cdots \circ \rho^k}^{m}(n).$$

(14) For all $k \ge 1$, $N(n)|\rho^k(n)$.

(15) For all m, $\overbrace{\rho_2 \circ \cdots \circ \rho_2}^{m} = \rho_{2^m}$.

Proof of (15): Note that for all $\alpha \geq 1$,

$$\lfloor \frac{\lfloor \frac{\alpha}{2} \rfloor}{2} \rfloor = \lfloor \frac{\alpha}{4} \rfloor$$

so $\rho_2 \circ \rho_2 = \rho_4$. The property follows by induction replacing α by $\frac{\alpha}{2^{m-1}}$.

(16) For all m, $\overbrace{\rho^2 \circ \cdots \circ \rho^2}^{m} = \rho^{2^m}$.

(17) More generally, for all $k \geq 2$ and $m \in \mathbb{N}$,

$$\overbrace{\rho^k \circ \cdots \circ \rho^k}^{m} = \rho^{k^m} \text{ and } \overbrace{\rho_k \circ \cdots \circ \rho_k}^{m} = \rho_{k^m}.$$

3. DIRICHLET SERIES AND EULER PRODUCTS

For integral $k \geq 1$ define the following Dirichlet series based on the lower and upper square roots:

$$\phi_k(s) = \sum_{n=1}^{\infty} \frac{\rho_k(n)}{n^s}, \, \phi^k(s) = \sum_{n=1}^{\infty} \frac{\rho^k(n)}{n^s}$$

The trivial case has $\phi_1(s) = \phi^1(s) = \zeta(s-1)$.

In [3] the case $k = 2$ was studied leading to the forms:

$$\phi_2(s) = \frac{\zeta(2s-1)\zeta(s)}{\zeta(2s)} \ \sigma > 1, \phi^2(s) = \frac{\zeta(2s-1)\zeta(s-1)}{\zeta(2s-2)} \ \sigma > 2$$

Since $N(n) = \sum_{d|n} \mu^2(d)\phi(d)$ we have

$$\begin{aligned} \phi_N(s) &= \zeta(s) \sum_{n=1}^{\infty} \frac{\mu^2(n)\phi(n)}{n^s} \\ &= \zeta(s) \prod_p (1 + \frac{\phi(p)}{p^s}) \\ &= \zeta(s) \prod_p \big(1 - \frac{1}{p^s} + \frac{1}{p^{s-1}}\big). \end{aligned}$$

This series converges absolutely for $\sigma > 2$.

Theorem 3.1. *For all $k > 1$ and $\sigma > 1$*

$$\phi_k(s) = \frac{\zeta(s)\zeta(ks-1)}{\zeta(ks)}.$$

Proof. By definition

$$\phi_k(s) = \sum_{n=1}^{\infty} \frac{\rho_k(n)}{n^s}$$

and therefore

$$\frac{\phi_k(s)}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{v_k(n)}{n^s}$$

where

$$v_k(n) = \sum_{d|n} \rho_k(d)\mu(\frac{n}{d}).$$

Now for each prime power

$$v_k(p^\alpha) = \rho_k(p^\alpha)\mu(1) + \rho_k(p^{\alpha-1})\mu(p) + 0 = p^{\lfloor \frac{\alpha}{k} \rfloor} - p^{\lfloor \frac{\alpha-1}{k} \rfloor}.$$

Therefore, if $k \nmid \alpha$, $v_k(p^\alpha) = 0$ whereas if $k \mid \alpha$ and $\alpha > 0$,

$$v_k(p^\alpha) = p^{\frac{\alpha}{k}} - p^{\frac{\alpha}{k}-1} = p^{\frac{\alpha}{k}}(1 - \frac{1}{p}).$$

Finally $v_k(1) = 1$.

From this it follows that if for some prime p with $p^\alpha \| n$, $k \nmid \alpha$, then $v_k(n) = 0$. Otherwise $v_k(n) = \prod_{p^\alpha \| n} p^{\frac{\alpha}{k}}(1 - \frac{1}{p})$. In this case n has a k'th root and we can write $v_k(n) = n^{\frac{1}{k}} \frac{\phi(n)}{n}$ where ϕ is Euler's totient function.

If we define $\omega_k(n) = 1$ if n has a k'th root and 0 otherwise, then

$$\begin{aligned}
\frac{\phi_k(s)}{\zeta(s)} &= \sum_{n=1}^{\infty} \frac{n^{\frac{1}{k}} \frac{\phi(n)}{n} \omega_k(n)}{n^s} \\
&= \sum_{m=1}^{\infty} \frac{m\phi(m^k)/m^k}{m^{ks}} \\
&= \sum_{m=1}^{\infty} \frac{m^k\phi(m)/m}{m^{ks+k-1}} \\
&= \sum_{m=1}^{\infty} \frac{\phi(m)}{m^{ks}} \\
&= \frac{\zeta(ks-1)}{\zeta(ks)}
\end{aligned}$$

and the given formula for $\phi_k(s)$ follows directly. ∎

Note that the result also holds for $k = 1$ but in the range $\sigma > 2$. Note also that the formula gives an analytic continuation of the function defined by the lower k'th root Dirichlet series to meromorphic functions on $\mathbb{C}$.

THEOREM 3.2. *For $\sigma > 2$*

$$\begin{aligned}\phi^k(s) &= \zeta(s)\zeta(ks-1)\prod_p(1-\frac{1}{p^s}+\frac{1}{p^{s-1}}-\frac{1}{p^{sk-1}})\\ &= \sum_{n=1}^{\infty}\frac{\mu(n)^2\phi(n)}{n^s}\prod_{p|n}(1-\frac{1}{p^{ks-1}})^{-1}.\end{aligned}$$

Proof. To derive the first expression let $v^k(n) = \sum_{d|n}\rho^k(d)\mu(\frac{n}{d})$.
and $\alpha = k\beta+1$ with $\beta = 0,1,\ldots,$ then $v^k(p^\alpha) = p^{\frac{\alpha-1}{k}}(p-1)$, and if $k\nmid\alpha-1$ then $v^k(p^\alpha)=0$. Therefore

$$\begin{aligned}\frac{\phi^k(s)}{\zeta(s)} &= \prod_p\big(1+\sum_{\beta=0}^{\infty}\frac{p^\beta(p-1)}{p^{(k\beta+1)s}}\big)\\ &= \prod_p\big(1+\frac{(p-1)}{p^s}\sum_{\beta=0}^{\infty}\frac{1}{p^{(ks-1)\beta}}\big)\\ &= \prod_p\big(1+\frac{(p-1)}{p^s}\cdot\frac{1}{1-\frac{1}{p^{ks-1}}}\big)\\ &= \zeta(ks-1)\prod_p\big(1-\frac{1}{p^{ks-1}}+\frac{p-1}{p^s}\big)\\ &= \zeta(ks-1)\prod_p\big(1-\frac{1}{p^s}+\frac{1}{p^{s-1}}-\frac{1}{p^{ks-1}}\big)\end{aligned}$$

and the first expression for $\phi^k(s)$ follows.
If for all $p\mid n$, $k\mid\alpha-1$ we have

$$\begin{aligned}v^k(n) &= \prod_{p^\alpha\|n}p^{\frac{\alpha-1}{k}}(1-\frac{1}{p})N(n)\\ &= \big(\frac{n}{N(n)}\big)^{\frac{1}{k}}\frac{\phi(n)}{n}N(n)\\ &= \big(\frac{n}{N(n)}\big)^{\frac{1}{k}-1}\phi(n).\end{aligned}$$

Hence

$$\begin{aligned}
\frac{\phi^k(s)}{\zeta(s)} &= \sum_{n=1}^{\infty} \frac{\left(\frac{n}{N(n)}\right)^{\frac{1}{k}-1}\phi(n)}{n^s}\omega_k(n/N(n)) \\
&= \sum_{\substack{a=1 \\ a \text{ squarefree}}}^{\infty} \sum_{\substack{b=1 \\ N(b)|a}}^{\infty} \frac{(b^k)^{\frac{1}{k}-1}\phi(ab^k)}{a^s b^k s} \text{ (using } n = ab^k) \\
&= \sum_{\substack{a=1 \\ a \text{ squarefree}}}^{\infty} \sum_{\substack{b=1 \\ N(b)|a}}^{\infty} \frac{bb^{-k}b^k\phi(a)}{a^s b^k s} \\
&= \sum_{\substack{a=1 \\ a \text{ squarefree}}}^{\infty} \frac{\phi(a)}{a^s} \sum_{\substack{b=1 \\ N(b)|a}}^{\infty} \frac{1}{b^{ks-1}} \\
&= \sum_{\substack{a=1 \\ a \text{ squarefree}}}^{\infty} \frac{\phi(a)}{a^s} \prod_{p|a} \left(1 - \frac{1}{p^{ks-1}}\right)^{-1}
\end{aligned}$$

and the expression given in the second part of the theorem statement follows directly from this. ▮

4. PARTIAL SUMS OF THE DIRICHLET SERIES

THEOREM 4.1. *Let* $R_k(x) = \sum_{n \le x} \rho_k(n)$. *Then*

$$R_k(x) = \frac{\zeta(k-1)}{\zeta(k)}x + O(x^{\frac{2}{k}}) \text{ for } k > 2 \text{ and } R_2(x) = \frac{x \log x}{\zeta(2)} + O(x).$$

Proof. Let $k > 2$ be in $\mathbb{N}$ and let $Q_k(x) = \#\{n \le x \mid n \text{ is k-free}\}$. Then [5] $Q_k(x) = \frac{x}{\zeta(k)} + O(x^{\frac{1}{k}})$. Then

$$\begin{aligned}
R_k(x) &= \sum_{d^k \le x} \sum_{\substack{\rho_k(n)=d \\ n \le x}} r_k(n) \\
&= \sum_{1 \le d \le x^{\frac{1}{k}}} d\#\{n \le x \mid r_k(n) = d\} \\
&= \sum_{1 \le d \le x^{\frac{1}{k}}} dQ_k(\frac{x}{d^k}) \\
&= \sum_{1 \le d \le x^{\frac{1}{k}}} d\big[\frac{x}{d^k\zeta(k)} + O(\frac{x^{\frac{1}{k}}}{d})\big] \\
&= \frac{x}{\zeta(k)} \sum_{1 \le d \le x^{\frac{1}{k}}} \frac{1}{d^{k-1}} + O(x^{\frac{2}{k}}) \quad (1) \\
&= \frac{x}{\zeta(k)}\big[\zeta(k-1) + \frac{(x^{1/k})^{1-(k-1)}}{1-(k-1)} + O(\frac{1}{x^{k-1}})\big] + O(x^{\frac{2}{k}}) \\
&= \frac{\zeta(k-1)}{\zeta(k)}x + \frac{x^{\frac{2}{k}}}{(2-k)\zeta(k)} + O(x^{2-k}) + O(x^{\frac{2}{k}}) \\
&= \frac{\zeta(k-1)}{\zeta(k)}x + O(x^{\frac{2}{k}}).
\end{aligned}$$

If $k = 2$, we start the derivation at line (1) above:

$$\begin{aligned}
R_2(x) &= \frac{x}{\zeta(2)} \sum_{1 \le x \le \sqrt{x}} \frac{1}{n} + O(x) \\
&= \frac{x}{\zeta(2)}[\log x + \gamma + O(\frac{1}{x})] + O(x) \\
&= \frac{x \log x}{\zeta(2)} + O(x)
\end{aligned}$$

∎

PROPOSITION 4.1. *The partial sums of the upper k'th root are given by*

$$\sum_{n \le x} \rho^k(n) = \sum_{n \le x} nd(n^k, k, \frac{n^k}{x})$$

where $d(n, k, \alpha) = \#\{d \mid 1 \le d \le n, d|n, d \text{ is k-free}, d \ge \alpha\}$. *for* $\alpha \ge 0$ *and* $k \ge 2$.

Proof. We rearrange the sum as follows:

$$\begin{aligned}\sum_{n\le x}\rho^k(n) &= \sum_{1\le b\le x}\sum_{\substack{\rho^k(n)=b\\ n\le x}} b\\ &= \sum_{1\le b\le x} b\#\{n\le x \mid \text{there exists k-free } a \text{ with } an=b^k\}\\ &= \sum_{1\le n\le x} nd(n^k,k,\frac{n^k}{x}).\end{aligned}$$

∎

Example 4.1. A nice formula can be obtained for the trivial case of the restricted divisor function which appears in Proposition 4.1, namely:

$$d(n^k,k,1)=k^{\omega(n)}$$

where $\omega(n)$ is the number of distinct prime factors dividing n. To see this note that if n has $m=\omega(n)$ and the prime factorization $n=\prod_{i=1}^m p^{\alpha_i}$ where each $\alpha_i\ge 1$, then $k\alpha_i\ge k$, so any divisor d of n^k which is k-free, will have the form $d=\prod_{i=1}^m p^{\beta_i}$ where for each i, $0\le\beta_i<k$. Conversely each such d is k-free. The formula follows directly from these observations.

In case $k=2$, $d(n^2,2,\alpha)=d(n,2,\alpha)$.

Lemma 4.1. *The partial sums of the upper k'th root are given by*

$$\sum_{n\le x}\rho^k(n)=\sum_{1\le n\le x^{\frac{1}{k}}} nN_k(\frac{x}{n^k}) \text{ where } N_k(x)=\sum_{\substack{n\le x\\ n \text{ is k-free}}} N(n).$$

Proof. We rearrange the sum as follows:

$$\begin{aligned}\sum_{n\le x}\rho^k(n) &= \sum_{n\le x} N(a)\rho_k(n) \text{ where } n=ab^k,\ a \text{ k-free}\\ &= \sum_{1\le b\le x^{\frac{1}{k}}} b\sum_{\substack{a\le\frac{x}{b^k}\\ a \text{ k-free}}} N(a)\\ &= \sum_{1\le n\le x^{\frac{1}{k}}} nN_k(\frac{x}{n^k}).\end{aligned}$$

∎

If a is a positive integer and $x>0$ define $T_a(x)=\sum_{1\le n\le x} N(an)$. Then [4] $T_1(x)=\frac{\alpha}{2}x^2+O(x^{3/2})$ where $\alpha=\prod_p(1-\frac{1}{p(p+1)})$.

Lemma 4.2.

$$N_k(x) = \sum_{1\le d\le x^{1/k}} \mu(d)T_d(\frac{x}{d^k})$$

Proof.

$$\begin{aligned} N_k(x) &= \sum_{\substack{n\le x \\ n \text{ k-free}}} N(n) \\ &= \sum_{n\le x} N(n)\sum_{d^k|n}\mu(d) \\ &= \sum_{1\le d\le x^{\frac{1}{k}}} \mu(d)\sum_{a\le x/d^k} N(ad) \\ &= \sum_{1\le d\le x^{\frac{1}{k}}} \mu(d)T_d(\frac{x}{d^k}) \end{aligned}$$

∎

Lemma 4.3. *For every prime integer* p *and square free* u *with* $(u,p)=1$*:*

$$T_u(x) = \frac{1}{p}T_{up}(x) + (1-\frac{1}{p})T_{up}(\frac{x}{p})$$

Proof.

$$\begin{aligned} T_{up}(x) &= \sum_{\substack{n\le x \\ p|n}} N(upn) + \sum_{\substack{n\le x \\ p\nmid n}} N(upn) \\ &= \sum_{\substack{n\le x \\ p|n}} N(un) + p\sum_{\substack{n\le x \\ p\nmid n}} N(un) \\ &= \sum_{m\le x/p} N(upm) + p[T_u(x) - \sum_{\substack{n\le x \\ p|n}} N(un)] \\ &= T_{up}(\frac{x}{p}) + p[T_u(x) - T_{up}(\frac{x}{p})] \end{aligned}$$

and the result follows directly on rearranging this formula. ∎

By setting $u=1$ in the above Lemma we obtain:

Corollary 4.1. *For every prime integer* p*:*

$$T_1(x) = \frac{1}{p}T_p(x) + (1-\frac{1}{p})T_p(\frac{x}{p}).$$

Example 4.2. Let p and q be distinct primes. Then

$$T_{pq}(x) = T_{pq}(\frac{x}{p}) + T_{pq}(\frac{x}{q}) - T_{pq}(\frac{x}{pq}) + pq[T_1(x) - T_p(\frac{x}{p}) - T_q(\frac{x}{q}) + T_{pq}(\frac{x}{pq}).$$

To see this write

$$T_{pq}(x) = \sum_{\substack{n\le x\\ p|n}} N(pqn) + \sum_{\substack{n\le x\\ q|n}} N(pqn) - \sum_{\substack{n\le x\\ p|n, q|n}} N(pqn) + \sum_{\substack{n\le x\\ p\nmid n, q\nmid n}} N(pqn)$$

and simplify.

Theorem 4.2. *Let the integer u be square free. Then for all $x \geq 1$:*

$$\sum_{d|u} \mu(d) T_u(\frac{x}{d}) = u \sum_{d|u} \mu(d) T_d(\frac{x}{d})$$

Proof. Express u as the product of distinct primes, $u = p_1 p_2 \dots p_m$. Expand $T_u(x)$:

$$\begin{aligned}
T_{p_1\dots p_m}(x) &= \sum_{n\le x} N(p_1\dots p_m n)\\
&= \sum_{p_i}\sum_{\substack{n\le x\\ p_i|n}} N(p_1\dots p_m n) - \sum_{p_{i_1}<p_{i_2}} \sum_{\substack{n\le x\\ p_{i_1}|n, p_{i_2}|n}} N(p_1\dots p_m n) + \dots\\
&\quad + \sum_{\substack{n\le x\\ p_1\nmid n,\dots,p_m\nmid n}} N(p_1\dots p_m n)\\
&= \sum_{p_i}\sum_{\substack{n\le x\\ p_i|n}} N(\frac{p_1\dots p_m}{p_i} n) - \sum_{p_{i_1}<p_{i_2}} \sum_{\substack{n\le x\\ p_{i_1}|n, p_{i_2}|n}} N(\frac{p_1\dots p_m}{p_{i_1}p_{i_2}} n) + \dots\\
&\quad + p_1\dots p_m \sum_{\substack{n\le x\\ p_1\nmid n,\dots,p_m\nmid n}} N(n)\\
&= \sum_{p_i} T_u(\frac{x}{p_i}) - \sum_{p_{i_1}<p_{i_2}} T_u(\frac{x}{p_{i_1}p_{i_2}}) + \dots\\
&\quad + p_1\dots p_m[\sum_{n\le x} N(n) - \sum_{p_i}\sum_{\substack{n\le x\\ p_i|n}} N(n) + \dots]
\end{aligned}$$

This enables us to write

$$T_u(x) = \sum_{p_i} T_u(\frac{x}{p_i}) - \sum_{p_{i_1}<p_{i_2}} T_u(\frac{x}{p_{i_1}p_{i_2}}) + \dots + u[T_1(x) - \sum_{p_i} T_{p_i}(\frac{x}{p_i}) + \dots]$$

and the formula follows from this after bringing the terms without the factor u onto the left hand side and using the fact that u is square free. ▮

THEOREM 4.3. *If there exists a function $\beta(u)$ such that for all square free u, $T_u(x) \sim \frac{\alpha}{2}\beta(u)x^2$ as $x \to \infty$, then*

$$\beta(u) = \prod_{p|u} \frac{p^3}{(p^2+p-1)}$$

Proof. By Corollary 4.1,

$$T_1(x) = \frac{1}{p}T_p(x) + (1-\frac{1}{p})T_p(\frac{x}{p})$$

Hence $$x^2 = \frac{1}{p}\beta(p)x^2 + (1-\frac{1}{p})\beta(p)(\frac{x^2}{p^2}) = x^2\beta(p)[\frac{1}{p} + (1-\frac{1}{p})\frac{1}{p^2}]$$

Therefore $$\beta(p) = \frac{p^3}{p^2+p-1}.$$

Now assume the expression for β is true when u is the product of n distinct primes $u = p_1 \dots p_n$, and consider the square free number up_{n+1}. Let $p = p_{n+1}$. By Lemma 4.3,

$$T_u(x) = \frac{1}{p}T_{up}(x) + (1-\frac{1}{p})T_{up}(\frac{x}{p})$$

Therefore $$\prod_{i=1}^{n} \frac{p_i^3}{p_i^2+p_i-1} = \frac{1}{p}\beta(up) + (1-\frac{1}{p})\frac{\beta(up)}{p^2}$$

Hence $$\beta(up) = (\prod_{i=1}^{n} \frac{p_i^3}{p_i^2+p_i-1})(\frac{p^3}{p^2+p-1})$$

and the formula given in the statement of the theorem follows by induction on n. ▮

THEOREM 4.4. *For each fixed square free $u \in \mathbb{N}$,*

$$T_u(x) \sim \frac{\alpha}{2}\beta(u)x^2$$

as $x \to \infty$, where $\beta(u)$ is defined in the previous theorem.

Proof. First note, using property (1), that $N(un)/u$ is multiplicative in n. Therefore

$$\begin{aligned}
\frac{1}{u}\cdot\sum_{n=1}^{\infty}\frac{N(un)}{n^s} &= \prod_p \left(1+\frac{N(up)}{up^s}+\frac{N(up^2)}{up^{2s}}+\cdots\right)\\
&= \prod_{p|u}\left(1+\frac{u}{up^s}+\frac{u}{up^{2s}}+\cdots\right)\times\prod_{p\nmid u}\left(1+\frac{up}{up^s}+\frac{up}{up^{2s}}+\cdots\right)\\
&= \prod_p\left(1-\frac{1}{p^s}\right)^{-1}\cdot\frac{\prod_p\left(1-\frac{1}{p^s}+\frac{1}{p^{s-1}}\right)}{\prod_{p|u}\left(1-\frac{1}{p^s}+\frac{1}{p^{s-1}}\right)}\\
&= \frac{1}{\prod_{p|u}\left(1-\frac{1}{p^s}+\frac{1}{p^{s-1}}\right)}\cdot\phi_N(s).
\end{aligned}$$

Hence

$$\sum_{n=1}^{\infty}\frac{N(un)}{n^s}=\prod_{p|u}\frac{p}{\left(1-\frac{1}{p^s}+\frac{1}{p^{s-1}}\right)}\cdot\phi_N(s)$$

This Dirichlet Series has a simple pole at $s=2$ with residue

$$\prod_{p|u}\frac{p}{\left(1-\frac{1}{p^2}+\frac{1}{p}\right)}\lim_{s\to 2}(s-2)\phi_N(s)$$

which is just

$$\beta(u)\lim_{s\to 2}(s-2)\phi_N(s).$$

It follows, for example from [9], given that the leading term for the partial sums $\sum_{n\le x}N(n)$ is $\frac{\alpha}{2}x^2$, that the leading term for the sums $\sum_{n\le x}N(un)$ is $\frac{\alpha}{2}\beta(u)x^2$. ∎

For each positive integer k and real $x\ge 1$ define the partial sum function

$$A_k(x)=\sum_{n\le x}\frac{\mu(n)\beta(n)}{n^k}.$$

Below an asymptotic expression for the partial sums of the upper k'th roots is derived in terms of A_k and then an expression for A_k as an infinite product.

Theorem 4.5. *The leading asymptotic order of the partial sums of the upper square roots are given by:*

$$\sum_{n\le x}\rho^k(n)\sim\frac{\alpha}{2}\zeta(2k-1)A_{2k}(x^{1/k})x^2$$

Proof.

$$\begin{aligned}
\sum_{n\le x}\rho^k(n) &= \sum_{1\le n\le x^{1/k}} nN_k(\frac{x}{n^k}) \text{ by Lemma 4.1}\\
&= \sum_{1\le n\le x^{1/k}} n \sum_{1\le d\le x^{1/k}/n} \mu(d)T_d(\frac{x}{(dn)^k}) \text{ by Lemma 4.2}\\
&\sim \frac{\alpha}{2}x^2\cdot S \text{ where the sum } S \text{ is given by}\\
S &= \sum_{1\le n\le x^{1/k}} \frac{1}{n^{2k-1}} \sum_{1\le d\le x^{1/k}/n} \frac{\mu(d)}{d^{2k}}\beta(d) \text{ Theorem 4.4}\\
&= \sum_{nd\le x^{1/k}} \frac{1}{n^{2k-1}}\cdot\frac{\mu(d)\beta(d)}{d^{2k}}.
\end{aligned}$$

Note that we can set $\beta(d)=0$ here if d is not square free.

Now let $F(x)=\sum_{n\le x} n^{1-2k}$ and apply partial summation [1] to get

$$S=\sum_{n\le x^{1/k}}\frac{\mu(n)\beta(n)}{n^{2k}}F(\frac{x^{1/k}}{n})$$

By [1] we can write

$$F(\frac{x^{1/k}}{n})=\zeta(2k-1)-\frac{x^{\frac{2}{k}-2}}{n^{2-2k}(2k-2)}+O(\frac{x^{\frac{1}{k}-2}}{n^{1-2k}})$$

Note that if we let $\beta(n)$ be the completely multiplicative extension of β, then it is easy to derive the bounds

$$n(4/5)^{\Omega(n)}\le\beta(n)\le n$$

where $\Omega(n)$ is the total number of prime factors of n. Also

$$|\sum_{n\le x^{1/k}}\frac{\mu(n)\beta(n)}{n}|\le\sum_{n\le x^{1/k}}\frac{\beta(n)}{n}=O(x^{1/k})$$

and

$$|\sum_{n\le x^{1/k}}\frac{\mu(n)\beta(n)}{n^2}|\le\sum_{n\le x^{1/k}}\frac{\beta(n)}{n^2}=O(\log x).$$

Therefore

$$\frac{\alpha x^2}{2}\cdot S=\frac{\alpha}{2}x^2\zeta(2k-1)A_{2k}(x^{1/k})+O(x^{2/k}\log x)+O(x^{2/k})$$

and the formula given in the theorem statement follows. ∎

THEOREM 4.6. *If* $k\ge3$

$$A_k(x)=\prod_{p\le x}\big(1-\frac{1}{p^{k-1}(p^2+p-1)}\big)+o(1).$$

Proof. This is a straight forward application of the theorem of Lucht [8]. Define a multiplicative arithmetical function f by $f(n) = \beta(n)n^{1-k}$. Then, for all primes p

$$f(p) = \frac{1}{p^{k-4}(p^2 + p - 1)}$$

so for $k \geq 2$ we have $0 < f(p) < 1$.

Consider the summation, for $k > 2$,

$$\sum_{p \leq x} \frac{f(p)\log p}{p} = \sum_{p \leq x} \frac{\log p}{p^{k-3}(p^2 + p - 1)} < \sum_{p \leq x} \frac{1}{p^{k-2}} < \sum_{n \leq x} \frac{1}{n^{k-2}} < \infty.$$

For these values of k the hypothesis of Lucht's theorem is satisfied, i.e.

$$\sum_{p \leq x} \frac{f(p)\log p}{p} = o(\log x)$$

Hence

$$A_k(x) = \prod_{p \leq x} \big(1 - \frac{1}{p^{k-1}(p^2 + p - 1)}\big) + o(1).$$

∎

COROLLARY 4.2. *The limit* $\lim_{x\to\infty} A_k(x) = \alpha_k$ *exists as a finite real number for every* $k \in \mathbb{N}$.

COROLLARY 4.3. *The partial sums of the upper k'th roots are given by:*

$$\sum_{n \leq x} \rho^k(n) \sim \frac{\alpha}{2}\zeta(2k-1)\alpha_{2k}x^2.$$

In case $k = 2$ we are able to derive the asymptotic order of the error of the partial sums of the upper square roots:

THEOREM 4.7.

$$\sum_{n \leq x} \rho^2(n) = \frac{x^2\zeta(3)}{2\zeta(2)} + O(x^{\frac{3}{2}})$$

Proof. First we derive an asymptotic expression for $N_2(x)$:

$$\begin{aligned}
N_2(x) &= \sum_{\substack{n\le x \\ n \text{ is square-free}}} N(n) \\
&= \sum_{\substack{n\le x \\ n \text{ is square-free}}} n \\
&= \sum_{n\le x} n \sum_{d^2|n} \mu(d) \\
&= \sum_{d\le\sqrt{x}} \mu(d)d^2 \sum_{m\le x/d^2} m \\
&= \sum_{d\le\sqrt{x}} \mu(d)d^2 \lfloor \frac{x}{d^2} \rfloor (\lfloor \frac{x}{d^2} \rfloor + 1) \\
&= \sum_{d\le\sqrt{x}} \mu(d)d^2 (\frac{x}{d^2} + O(1))^2 \\
&= \frac{x^2}{2} \sum_{d\le\sqrt{x}} \mu(d)d^2 \cdot \frac{1}{d^4} + O(x \sum_{d\le\sqrt{x}} \frac{|\mu(d)|d^2}{d^2}) \\
&= \frac{x^2}{2} \sum_{d\le\sqrt{x}} \frac{\mu(d)}{d^2} + O(x^{\frac{3}{2}}) \\
&= \frac{x^2}{2\zeta(2)} + O(x^{\frac{3}{2}})
\end{aligned}$$

Next this expression is substituted in the second partial sum formula from the proposition above:

$$\begin{aligned}
\sum_{n\le x} \rho^2(n) &= \sum_{1\le n\le\sqrt{x}} nN_2(\frac{x}{n^2}) \\
&= \sum_{1\le n\le\sqrt{x}} n[\frac{x^2}{n^4 2\zeta(2)} + O(\frac{x^{3/2}}{n^3}) \\
&= \frac{x^2}{2\zeta(2)} \sum_{1\le n\le\sqrt{x}} \frac{1}{n^3} + O(x^{3/2} \sum_{1\le n\le\sqrt{x}} \frac{1}{n^2}) \\
&= \frac{x^2\zeta(3)}{2\zeta(2)} + O(x^{\frac{3}{2}}).
\end{aligned}$$

∎

The asymptotic order for the partial sums of the upper k'th roots are all it appears very close approximations to the partial sums for the conductor, a well studied problem, [4, 11]. Indeed, even the (upper) square root, appears to lead to a good approximation for this sum. By $x = 400$ the relative difference is less than 4%.

Finally, if we define a subsequence Dirichlet series ϕ_{N_k} corresponding to that of the conductor $N(n)$, but summing only over k-free integers, we obtain an interesting relationship with the Dirichlet series of the upper k'th root:

$$
\begin{aligned}
\phi_{N_k}(s) &= \sum_{n=1, n \text{ is k-free}}^{\infty} \frac{N(n)}{n^s} \\
&= \prod_p \big(1 + \frac{N(p)}{p^s} + \cdots + \frac{N(p^{k-1})}{p^{s(k-1)}}\big) \\
&= \prod_p \big(1 + \frac{p}{p^s}[1 + \frac{1}{p^s} + \cdots + \frac{1}{p^{s(k-2)}}]\big) \\
&= \prod_p \big(1 + \frac{p}{p^s} \cdot \frac{1 - \frac{1}{p^{s(k-1)}}}{1 - \frac{1}{p^s}}\big) \\
&= \zeta(s) \prod_p \big(1 + \frac{1}{p^{s-1}} - \frac{1}{p^s} - \frac{1}{p^{ks-1}}\big) \\
&= \frac{\phi^k(s)}{\zeta(ks-1)} \text{ by Theorem 3.2.}
\end{aligned}
$$

5. LIMIT RELATIONSHIPS

PROPOSITION 5.1. *For all s with $\sigma > 2$:*

$$\lim_{k \to \infty} \phi^k(s) = \phi_N(s)$$

uniformly on each half plane with $\sigma \geq \sigma_0 > 2$.

Proof. Let $\sigma_0 > 2$ and $\epsilon > 0$ be given. Then there is an N_ϵ such that $\sum_{N+1}^{\infty} n^{1-\sigma} < \epsilon/2$ for all $N \geq N_\epsilon$ and $\sigma \geq \sigma_0$. Hence

$$
\begin{aligned}
\Big|\sum_{n=1}^{\infty} \frac{\rho^k(n)}{n^s} - \sum_{n=1}^{\infty} \frac{N(n)}{n^s}\Big| \leqslant{} & \Big|\sum_{n=1}^{N_\epsilon} \frac{\rho^k(n) - N(n)}{n^s}\Big| \\
& + \sum_{n=1+N_\epsilon}^{\infty} \frac{\rho^k(n)}{n^\sigma} + \sum_{n=1+N_\epsilon}^{\infty} \frac{N(n)}{n^\sigma}.
\end{aligned}
$$

By property (12) we can chose $k_0 \geq 2$ so that $\rho^k(n) = N(n)$ for all $k \geq k_0$ and for all n with $1 \leq n \leq N_\epsilon$. For these k

$$|\sum_{n=1}^{\infty} \frac{\rho^k(n)}{n^s} - \sum_{n=1}^{\infty} \frac{N(n)}{n^s}| \leqslant 2 \sum_{1+N_\epsilon}^{\infty} < \epsilon$$

so the limit result follows directly. ▮

Let $G_{\mathbb{Z}}$ be the semigroup of positive integers and let $\mathrm{Dir}(G_{\mathbb{Z}})$ be the Banach space of all functions represented by convergent Dirichlet series

$$f(s) = \sum_{n=1}^{\infty} \frac{f^{\#}(n)}{n^s}$$

on some half plane $\sigma > 0$ with norm $\|f\| = 1/ < f >$ where $< f >$ is $min\{j \mid f^{\#}(j) \neq 0\}$ or $\|f\| = 0$ if f is the zero Dirichlet series,[7].

Then if $*$ represents Dirichlet multiplication, $\|f * g\| \leq \|f\|\|g\|$ and $\mathrm{Dir}(G_{\mathbb{Z}})$ is a Banach algebra which is an integral domain.

PROPOSITION 5.2. *In Dir($G_{\mathbb{Z}}$), $\phi^k \to \phi_N$.*

Proof. This is immediate, since given N there is a k_0 such that $\rho^k(n) = \phi_N(n)$ for $1 \leq n \leq N$ and all $k \geq k_0$. This means $< \rho^k - \phi_N > \geqslant N + 1$. ▮

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