Climate Change Equations

In these pages the main focus is the various sets of equations including their variables, scales, coordinate systems and boundary conditions, for the study of geophysical flows. As well as the rather standard sets of equations, we include the simpler Boussinesq, geostrophic and quasi-geostrophic approximate models.

(1) Constants

$R := 287\ m^2/(s^2K)$ gas constant at normal temperatures and pressures
$R_{eq} = 6378.1366\ m$ equatorial radius of the earth in meters
$R_{po} = 6356.752\ m$ polar radius of the earth in meters
$R_{vo} = 6371.000\ m$ volumetric radius of the earth in meters
$e := 0.003353$ ellipticity or flattening of the spheroidal earth
$\Omega = 7.2921 \times 10^{-5}\ s^{-1}$ angular velocity of rotation in radians per second
$v_{eq} := 450.202 m/s$ earth speed at the equator
$g := 9.820\ m/s^2$ average surface gravity
$P := 23.9345\ hrs$ siderial rotation period $P = 2\pi/\Omega$ in seconds

$\mu = 1.729 \times 10^{-5}$ coefficient of dynamic viscosity air $0^o\ C$, $1\ Atm$ ($101325\ N/m^2$)
$\mu = 1.778 \times 10^{-5}$ coefficient of dynamic viscosity air $10^o\ C$, $1\ Atm$)
$v = 1.338 \times 10^{-5}$ kinematic viscosity $v = \nu/\rho_0$ air $0^o\ C$, $1\ Atm$
$v = 1.426 \times 10^{-5}$ kinematic viscosity $v = \nu/\rho_0$ air $0^o\ C$, $1\ Atm$
$\mu = 1.8 \times 10^{-6}$ coefficient of dynamic viscosity sea water $0^o\ C$ [talley p191]
$\mu = 1.0 \times 10^{-6}$ coefficient of dynamic viscosity sea water $20^o\ C$ [talley p191]

$\nu_E = 10^{-4} m^2/s$ molecular vertical eddy viscosity coefficient [roisin95]
$\nu_E = 1.83 \times 10^{-6} m^2/s$ at $S = 35$, $T = 0^oC$ [talley tbl S7.1]
$\nu_E = 1.05 \times 10^{-6} m^2/s$ at $S = 35$, $T = 20^oC$ [talley tbl S7.1]

$\kappa_E = 1.0 \times 10^{-4}$ average eddy diffusivity vertical coefficient
$\mathscr{A} \in [10^2, 10^4]$ range for the horizontal eddy or turbulent
viscosity coefficient [talley tbl S7.1]
$\kappa_S = 1.0 \times 10^{-2}$ salt diffusivity in sea water [pope]
$\kappa_S = 1.0 \times 10^{-2}$ salt molecular diffusivity in sea water [tally tbl S7.1]
$\kappa_q =$ molecular moisture diffusion coefficient
$\kappa_T = 1.37 \times 10^{-7}$ molecular heat kinematic diffusivity
at $S = 35$, $T = 0^oC$ [talley tbl S7.1]

$\kappa_T = 1.46 \times 10^{-7}$ molecular heat kinematic diffusivity
at $S = 35, T = 20^o C$ [talley tbl S7.1]

$\alpha = 1.7 \times 10^{-4}\ K^{-1}$ coefficient of sea water thermal expansion
$\beta = 7.6 \times 10^{-4}$ coefficient of saline contraction
$C_v := 718\ J/(kgK)$ heat capacity at constant volume for air at sea level
$C_v := 3990\ J/(kgK)$ heat capacity at constant volume for sea water

$p_0 :=$ reference pressure value for sea water
$\rho_0 := 1028\ kg/m^3$ reference density value for sea water
$T_0 := 283^o\ K$ reference temperature value for sea water
$S_0 := 35 psu$ reference sea water salinity value

Universal constants

Avagadro's number: $N_A = 6.02214076 \times 10^{23}$ the number of molecules in a mole.
Boltzman's constant: $k = 1.380649 \times 10^{-23}$ J/K which is R/N_A, where R the universal gas constant.

(2) Variables

$t :$ time normally in seconds, but alternative scales arise
$x, y :$ horizontal components Euclidian earth surface local coordinates
$z :$ vertical component of a Euclidian coordinate system
$u, v, w :$ Euclidian components of velocity being functions of t, x, y, z
$\lambda, \varphi :$ longitude and latitude $-\pi \leq \lambda < \pi, = -\pi/2 \leq \varphi \leq \pi/2$

$p :$ pressure in Newtons per square meter
$\rho :$ density in kilograms per cubic meter
$T :$ temperature in degrees Kelvin
$S :$ salinity in "practical salinity units" psu's
$q :$ specific humidity of air
$\lambda :$ longitude in radians

φ : latitude in radians

τ^{xy} : force per unit area on a cube on the x face in the y direction

f : the Coriolis parameter

f_* the reciprocal Coriolis parameter

where

$$q := \frac{\text{mass of water vapour}}{\text{mass of dry air} + \text{mass of water vapour}}$$

$$f := 2\Omega \sin\varphi \text{ the Coriolis parameter}$$

$$f_* := 2\Omega \cos\varphi$$

the conjugate Coriolis parameter

For dry air we have the ideal gas equation:

$$\rho = \frac{p}{RT} \text{ where } R \text{ is the ideal gas constant.}$$

Note however that if unconfined air is heated its pressure decreases. This is because the heating causes the space between molecules of air, on average, to increase, thus reducing the density and hence the pressure. Note also that pressure decreases exponentially with height as the density of air decreases due to a reduction in gravitational attraction. Under some circumstances we can write the approximation with P_0 a reference level and z the height above that level

$$P(z) = P_0 \exp\left(-\frac{mgz}{kT}\right),$$

where k is Boltzman's constant, T is temperature in degrees Kelvin (absolute 0 is -273.16 degrees C), and m is the average air molecule mass.
Another explanation: When anything is heated energy is imparted to its molecules in an eventually random manner. Their velocity and kinetic energy increases. If the material is confined collisions with the confining material will increase and become stronger. The Pressure increases. If the confining material is plastic or fluid the higher pressure does work against the confinement until the pressures equalize. If the confinement is the atmosphere the heated gas expands and becomes less dense. Density differences in a gravitational field result in convection which is experienced as wind.

For moist air:

$$\rho = \frac{p}{RT(1+0.608q)} \text{ where } q \text{ is the specific humidity of air.}$$

Note that moist air is less dense than dry air, since oxygen and nitrogen molecules are replaced by water molecules which are lighter. Also, sinking air (anticyclones) increases pressure at the surface because of an increase in density. Rising air (cyclones) decreases surface pressure.
For sea water:

$$\rho = \rho_0(1-\alpha(T-T_0)+\beta(S-S_0)) \text{ where the subscripted variables are reference values.}$$

The assumption that the fluid is Newtonian means at every point the viscous stresses are a linear function of the local strain rate, which is the time rate of change of the fluids deformation. In other words stresses are proportional to the rate of change of the velocity vector. This assumption, together with the continuity equation enables one to write, for Cartesian coordinates and μ the coefficient of dynamic viscosity:

$$\tau^{xx} = \mu\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial x}\right), \quad \tau^{xy} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right), \tau^{xz} = \mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right),$$

$$\tau^{yy} = \mu\left(\frac{\partial v}{\partial y} + \frac{\partial v}{\partial y}\right), \quad \tau^{yz} = \mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right),$$

$$\tau^{zz} = \mu\left(\frac{\partial w}{\partial z} + \frac{\partial w}{\partial z}\right),$$

with $\tau^{yx} = \tau^{xy}$, $\tau^{zx} = \tau^{xz}$ and $\tau^{zy} = \tau^{yz}$.

(3) Equations for air and sea water

Equations common to air and sea water:
These are the continuity, momentum and energy equations which take the following forms, using the so-called advective derivative which is defined by:

$$\frac{d}{dt} := \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}.$$

Here we are assuming there are no sources or sinks for the flow.

continuity:

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} + \frac{\partial \rho w}{\partial z} = 0,$$

x-momentum:

$$\rho\left(\frac{du}{dt} + f_* w - fv\right) = -\frac{\partial p}{\partial x} + \frac{\partial \tau^{xx}}{\partial x} + \frac{\partial \tau^{xy}}{\partial y} + \frac{\partial \tau^{xz}}{\partial z},$$

y-momentum:

$$\rho\left(\frac{dv}{dt} + fu\right) = -\frac{\partial p}{\partial y} + \frac{\partial \tau^{yx}}{\partial x} + \frac{\partial \tau^{yy}}{\partial y} + \frac{\partial \tau^{yz}}{\partial z},$$

z-momentum:

$$\rho\left(\frac{dw}{dt} - f_* * u\right) = -\frac{\partial p}{\partial z} - \rho g + \frac{\partial \tau^{zx}}{\partial x} + \frac{\partial \tau^{zy}}{\partial y} + \frac{\partial \tau^{zz}}{\partial z},$$

energy:

$$\rho C_v \frac{dT}{dt} + p\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) = k_T \nabla^2 T.$$

Equation of state for air as an ideal gas:
$\rho = \frac{p}{RT}$,
where R is the ideal gas constant.

Humidity equation for air:
$\frac{dq}{dt} = \kappa_q \nabla^2 q$,
where κ_q is the molecular moisture diffusion coefficient.

Equation of state for sea water:
$\rho = \rho_0 \left(1 - \alpha\left(T - T_0\right) + \beta\left(S - S_0\right)\right)$,

where $\rho_0,\ T_0,\ S_0$ are reference values for density, temperature and salinity respectively, α is the coefficient of thermal expansion and β is the coefficient of saline contraction.

Salt equation for sea water:

$\frac{dS}{dt} = \kappa_S \nabla^2 S,$

where k_S us the mollecular diffusivity of salt in sea water.

(4) The Boussinesq approximation:

This simplified set of equations is derived assuming ρ is constant other than in the buoyancy terms $\rho g z$, variations in temperature T are very small, the fluids are incompressible, the spacial derivatives of T are small, and the flow velocities u, v, w are also small. We have a closed system of five Cartesian equations in the variables u, v, w, p and ρ and parameters $\mathscr{A},\ \nu_E$ and k_E:

x-momentum:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} - fv =$$
$$-\frac{1}{\rho_0}\frac{\partial p}{\partial x} + \frac{\partial}{\partial x}\left(\mathscr{A}\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(\mathscr{A}\frac{\partial u}{\partial y}\right) + \frac{\partial}{\partial z}\left(\nu_E\frac{\partial u}{\partial z}\right),$$

y-momentum:

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} + fu =$$
$$-\frac{1}{\rho_0}\frac{\partial p}{\partial y} + \frac{\partial}{\partial x}\left(\mathscr{A}\frac{\partial v}{\partial x}\right) + \frac{\partial}{\partial y}\left(\mathscr{A}\frac{\partial v}{\partial y}\right) + \frac{\partial}{\partial z}\left(\nu_E\frac{\partial v}{\partial z}\right),$$

z-momentum:

$$-\frac{\partial p}{\partial z} - \rho g = 0,$$

continuity:

$$\frac{\partial u}{\partial z} + \frac{\partial v}{\partial z} + \frac{\partial w}{\partial z} = 0,$$

$Energy:$

$$\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + v\frac{\partial \rho}{\partial y} + w\frac{\partial \rho}{\partial z} =$$
$$\frac{\partial}{\partial x}\left(\mathscr{A}\frac{\partial \rho}{\partial x}\right) + \frac{\partial}{\partial y}\left(\mathscr{A}\frac{\partial \rho}{\partial y}\right) + \frac{\partial}{\partial z}\left(\kappa_E\frac{\partial \rho}{\partial x}\right).$$

Coriolis parameter:

$$f = 2\Omega \sin \varphi,$$
$$f_* = 2\Omega \cos \varphi.$$

Note that the Boussinesq approximation might not be valid in other coordinate systems, such as spherical or oblate spherical.

Acknowledgement: Chapter 3, "Introduction to geophysical fluid dynamics: physical and numerical aspects" 2nd edition, by Benoit Cushman-Roisin and Jean-Marie Beckers, AP, Elsevier, 2011.

See also: Chapter 4.

(5) Spherical Coordinates:

In these equations φ is latitude in radians with $-\pi/2 \leq \varphi \leq \pi/2$, λ longitude with $-\pi \leq \lambda < \pi$, and r in meters the distance from the center of the earth. In spherical coordinates the scale factors are given by $h_\varphi = r$, $h_\lambda = r\cos\varphi$, $h_r = 1$. The metric differential squared is

$$ds^2 = r^2 d\varphi^2 + r^2 \cos^2 \varphi \, d\lambda^2 + dr^2.$$

The material derivative then takes the form:

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \frac{u}{r\cos\varphi}\frac{\partial}{\partial \lambda} + \frac{v}{r}\frac{\partial}{\partial \varphi} + w\frac{\partial}{\partial r}.$$

In spherical coordinates, the equations which are common to air and sea water are:

$$\frac{\partial}{\partial t}(\rho\cos\varphi) + \frac{\partial}{\partial \lambda}\left(\frac{\rho u}{r}\right) + \frac{\partial}{\partial \varphi}\left(\frac{\rho v \cos\varphi}{r}\right) + \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \rho w \cos\varphi\right) = 0,$$

$$\rho\left(\frac{du}{dt} - \frac{uv\tan\varphi}{r} + \frac{uw}{r} + f_* w - fv\right) = -\frac{1}{r\cos\varphi}\frac{\partial p}{\partial \lambda} + F_\lambda,$$

$$\rho\left(\frac{dv}{dt} + \frac{u^2\tan\varphi}{r} + \frac{vw}{r} + f * u\right) = -\frac{1}{r}\frac{\partial p}{\partial \varphi} + F_\varphi,$$

$$\rho\left(\frac{dw}{dt} - \frac{u^2+v^2}{r} - f_* * u\right) = -\frac{\partial p}{\partial r} - \rho g + F_r,$$

$$\rho C_v \frac{dT}{dt} + p\left(\frac{\partial}{\partial \lambda}\left(\frac{\rho u}{r}\right) + \frac{\partial}{\partial \varphi}\left(\frac{\rho v\cos\varphi}{r}\right) + \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\rho w\cos\varphi\right)\right) = k_T \nabla^2 T,$$

where the stress terms $F_\lambda, F_\varphi, F_r$ are given by

$$F_\lambda = \mu\nabla^2 u, \;\; F_\varphi = \mu\nabla^2 v, \;\; F_r = \mu\nabla^2 w.$$

and where the Laplacian $\nabla^2 f(\lambda, \varphi, r)$ in spherical coordinates is

$$\nabla^2 f = \frac{1}{r^2\cos^2\varphi}\frac{\partial^2 f}{\partial\lambda^2} + \frac{1}{r^2\cos\varphi}\frac{\partial}{\varphi}\left(\cos\varphi\frac{\partial f}{\partial\varphi}\right) + \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right).$$

Finally the volume element is $dV = r^2\cos\varphi\ d\lambda d\varphi dr$.

Acknowledgement: Appendix A.2, "Introduction to geophysical fluid dynamics: physical and numerical aspects" 2nd edition, by Benoit Cushman-Roisin and Jean-Marie Beckers, AP, Elsevier, 2011.

See also: Sections 14.3, 14.4.

(6) Perturbed oblate spherical coordinates
Because global models often attempt to be as realistic as possible, and gravity is such an important factor in the equations, the earth and flows associated with it in those cases need to be approximated with not spherical but perturbed oblate spheroidal coordinates. These are denoted by the expression "geo-potential surfaces". First we approximate using oblate spheroidal coordinates (λ, φ, r). They are related to spherical polar coordinates denoted by $(\lambda, \varphi_o, r_o)$, using a constant $d = 521.854\ km$ which is equal to half the distance between the foci of the approximating ellipsoid of revolution,

$$r_o^2 = r^2 + \tfrac{1}{2}d^2 - d^2\sin^2\varphi,$$
$$r_o^2\cos^2\varphi_o = (r^2 + \tfrac{1}{2}d^2)\cos^2\varphi.$$

The ellipsoid of best fit to the sea-level geopotential surface has $r_0 = 6367.456\ km$ with parameters

$$\text{Semi-major axis:} \quad \sqrt{r_o^2 + \frac{1}{2}d^2} = 6378.139\ km,$$
$$\text{Semi-minor axis:} \quad \sqrt{r_o^2 - \frac{1}{2}d^2} = 6356.754\ km.$$

The scale factors are given by

$$
\begin{aligned}
h_\lambda^2 &= (r^2 + \tfrac{1}{2}d^2)\cos^2\varphi,\\
h_\varphi^2 &= r^2 - \tfrac{1}{2}d^2 + d^2\sin^2\varphi,\\
h_r^2 &= \frac{r^2(r^2 - \frac{1}{2}d^2 + d^2\sin^2\varphi)}{r^4 - d^4/4}.
\end{aligned}
$$

Using these approximations, when d is small the error is bounded by $d^2/(4r^2)$ which is less than 0.17%. Not only that but, apparently, the equations are the same as those written above for normal spherical coordinates!

(7) Reynolds averaging of the flow equations

We introduce an averaging operation $\langle X\rangle$ for expressions X defined in terms of flow variables. We write

$$X = \langle X\rangle + X',$$

so $X' = X - \langle X\rangle$ is the residual. This average could be a simple weighted sum or defined in terms of an integral operator over space and/or time. It must however satisfy three properties, where α, β are independent of the expressions X, Y which are being averaged:

$$
\begin{aligned}
\langle \alpha X + \beta Y\rangle &= \alpha\langle X\rangle + \beta\langle Y\rangle,\\
\langle\alpha\rangle &= \alpha,\\
\frac{\partial\langle X\rangle}{\partial t} &= \langle\frac{\partial X}{\partial t}\rangle.
\end{aligned}
$$

We also assume $\langle\partial X/\partial x\rangle = \partial\langle X\rangle/\partial x$, $\langle\partial X/\partial y\rangle = \partial\langle X\rangle/\partial y$, $\langle\partial X/\partial z\rangle = \partial\langle X\rangle/\partial z$. Hence for example

$$\langle X'\rangle = \langle X - \langle X\rangle\rangle = \langle X\rangle - \langle\langle X\rangle\rangle = \langle X\rangle - \langle X\rangle = 0.$$

We also have for expressions X, Y the “average product rule"

$$
\begin{aligned}
\langle XY\rangle &= \langle\langle X\rangle\langle Y\rangle\rangle + \langle\langle X\rangle Y'\rangle + \langle\langle Y\rangle X'\rangle + \langle X'Y'\rangle\\
&= \langle X\rangle\langle Y\rangle + \langle X'Y'\rangle.
\end{aligned}
$$

Averaging over the x-momentum equation with constant density ρ_0 and viscosity ν we get

$$\frac{\partial\langle u\rangle}{\partial t}+\frac{\partial\langle uu\rangle}{\partial x}+\frac{\partial\langle vu\rangle}{\partial y}+\frac{\partial\langle wu\rangle}{\partial z}+f_*\langle w\rangle-f\langle v\rangle=-\frac{1}{\rho_0}\frac{\partial\langle p\rangle}{\partial x}+\nu\nabla^2\langle u\rangle,$$

Using the average product rule we then get

$$\begin{aligned}&\frac{\partial\langle u\rangle}{\partial t}+\frac{\partial\langle u\rangle\langle u\rangle}{\partial x}+\frac{\partial\langle v\rangle\langle u\rangle}{\partial y}+\frac{\partial\langle w\rangle\langle u\rangle}{\partial z}+f_*\langle w\rangle-f\langle v\rangle\\&\qquad=-\frac{1}{\rho_0}\frac{\partial\langle p\rangle}{\partial x}+\nu\nabla^2\langle u\rangle-\frac{\partial\langle u'u'\rangle}{\partial x}-\frac{\partial\langle u'v'\rangle}{\partial x}-\frac{\partial\langle u'w'\rangle}{\partial x}.\end{aligned}$$

Expanding the Laplacian gives

$$\begin{aligned}&\frac{\partial\langle u\rangle}{\partial t}+\frac{\partial\langle u\rangle\langle u\rangle}{\partial x}+\frac{\partial\langle v\rangle\langle u\rangle}{\partial y}+\frac{\partial\langle w\rangle\langle u\rangle}{\partial z}+f_*\langle w\rangle-f\langle v\rangle+\frac{1}{\rho_0}\frac{\partial\langle p\rangle}{\partial x}\\&\quad=\frac{\partial}{\partial x}\left(\nu\frac{\partial\langle u\rangle}{\partial x}-\langle u'u'\rangle\right)+\frac{\partial}{\partial y}\left(\nu\frac{\partial\langle u\rangle}{\partial y}-\langle u'v'\rangle\right)+\frac{\partial}{\partial z}\left(\nu\frac{\partial\langle u\rangle}{\partial z}-\langle u'w'\rangle\right).\end{aligned}$$

Doing the same for the y-momentum equation we get

$$\begin{aligned}&\frac{\partial\langle v\rangle}{\partial t}+\frac{\partial\langle v\rangle\langle u\rangle}{\partial x}+\frac{\partial\langle v\rangle\langle v\rangle}{\partial y}+\frac{\partial\langle v\rangle\langle w\rangle}{\partial z}+f\langle u\rangle+\frac{1}{\rho_0}\frac{\partial\langle p\rangle}{\partial y}\\&\quad=\frac{\partial}{\partial x}\left(\nu\frac{\partial\langle v\rangle}{\partial x}-\langle v'u'\rangle\right)+\frac{\partial}{\partial y}\left(\nu\frac{\partial\langle v\rangle}{\partial y}-\langle v'v'\rangle\right)+\frac{\partial}{\partial z}\left(\nu\frac{\partial\langle v\rangle}{\partial z}-\langle v'w'\rangle\right).\end{aligned}$$

For the z-momentum:

$$\begin{aligned}&\frac{\partial\langle w\rangle}{\partial t}+\frac{\partial\langle w\rangle\langle u\rangle}{\partial x}+\frac{\partial\langle w\rangle\langle v\rangle}{\partial y}+\frac{\partial\langle w\rangle\langle w\rangle}{\partial z}-f_*\langle u\rangle+\frac{1}{\rho_0}\frac{\partial\langle p\rangle}{\partial z}\\&\quad=\frac{\partial}{\partial x}\left(\nu\frac{\partial\langle w\rangle}{\partial x}-\langle w'u'\rangle\right)+\frac{\partial}{\partial y}\left(\nu\frac{\partial\langle w\rangle}{\partial y}-\langle w'v'\rangle\right)+\frac{\partial}{\partial z}\left(\nu\frac{\partial\langle w\rangle}{\partial z}-\langle w'w'\rangle\right).\end{aligned}$$

Averaging the continuity equation gives

$$\frac{\partial\langle u\rangle}{\partial x}+\frac{\partial\langle v\rangle}{\partial y}+\frac{\partial\langle w\rangle}{\partial z}=0.$$

Hence, replacing the mean flow variables with actual variables and employing the standard Boussinesque assumptions we get the set of 5×5 set of equations in u, v, w, p and ρ. The eddy viscosity and diffusivity coefficients $\mathscr{A}, \nu_E$ and κ_E are either constant or functions of the flow variables and/or their derivatives and the grid structure. This is then called the **Boussinesq approximation** or **Boussinesq model** for a geophysical fluid:

x-momentum:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} - fv =$$
$$-\frac{1}{\rho_0}\frac{\partial p}{\partial x} + \frac{\partial}{\partial x}\left(\mathscr{A}\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(\mathscr{A}\frac{\partial u}{\partial y}\right) + \frac{\partial}{\partial z}\left(\nu_E\frac{\partial u}{\partial x}\right),$$

y-momentum:

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} + fu =$$
$$-\frac{1}{\rho_0}\frac{\partial p}{\partial x} + \frac{\partial}{\partial x}\left(\mathscr{A}\frac{\partial v}{\partial x}\right) + \frac{\partial}{\partial y}\left(\mathscr{A}\frac{\partial v}{\partial y}\right) + \frac{\partial}{\partial z}\left(\nu_E\frac{\partial v}{\partial x}\right),$$

z-momentum:

$$-\frac{\partial p}{\partial z} - \rho g = 0,$$

continuity:

$$\frac{\partial u}{\partial z} + \frac{\partial v}{\partial z} + \frac{\partial w}{\partial z} = 0,$$

energy:

$$\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + v\frac{\partial \rho}{\partial y} + w\frac{\partial \rho}{\partial z} =$$
$$-\frac{1}{\rho_0}\frac{\partial p}{\partial x} + \frac{\partial}{\partial x}\left(\mathscr{A}\frac{\partial \rho}{\partial x}\right) + \frac{\partial}{\partial y}\left(\mathscr{A}\frac{\partial \rho}{\partial y}\right) + \frac{\partial}{\partial z}\left(\kappa_E\frac{\partial \rho}{\partial x}\right).$$

Acknowledgement: Sections 4.1, 4.2 "Introduction to geophysical fluid dynamics: physical and numerical aspects" 2nd edition, by Benoit Cushman-Roisin and Jean-Marie Beckers, AP, Elsevier, 2011.

See also: Sections 14.3, 14.4.

(8) Earth System Models of Intermediate Complexity

We give the details for only one of a wide range of different models, and only for the atmosphere. Here the dimensionality is reduced by averaging over latitude, the y variable. If $\langle x\rangle$ is the average then we write as usual $x = \langle x\rangle + x'$. The averaging must be linear over constants and commute with differentiation. Let u, v, w be the flow values in the x, y, z directions respectively with z vertical, x pointing to the south pole and y directly east making a right handed system. Let also Q be the heating as a function of latitude, Φ friction, R the gas constant for dry air, c_p specific heat of air at constant pressure, $f = 2\Omega \sin\varphi$ the Coriolis parameter. Averaging over the ideal gas equation we get

$$\langle\rho\rangle = \frac{\langle p\rangle}{R\langle T\rangle}.$$

We have the following equations:
Zonal (average over changing latitude) momentum:

$$\frac{\partial\langle v\rangle}{\partial t} - f\langle u\rangle = -\frac{\partial(\langle u'v'\rangle)}{\partial x} + \Phi.$$

Meridional momentum (geostrophic balance):

$$f\langle v\rangle + R\langle T\rangle\frac{\partial}{\partial\lambda}(\ln\langle p\rangle) = 0.$$

Vertical hydrostatic balance:

$$\frac{\langle p\rangle}{\partial z} = -\frac{\langle p\rangle g}{R\langle T\rangle}.$$

Thermodynamic balance:

$$\frac{\partial\langle T\rangle}{\partial t} + \frac{\partial\langle u'T'\rangle}{\partial x} + \frac{\partial\langle w'T'\rangle}{\partial z} + \langle w\rangle\left(\frac{g}{\langle\rho\rangle c_p} + \frac{\partial\langle T\rangle}{\partial z}\right) = \frac{Q}{\langle\rho\rangle c_p}.$$

Mass balance (continuity):

$$\frac{\partial(\langle\rho\rangle\langle u\rangle)}{\partial x} + \frac{\partial(\langle\rho\rangle\langle w\rangle)}{\partial z} = 0.$$

To close this set of equations we need values for $\langle u'T'\rangle$ and $\langle u'v'\rangle$ in terms of the univalent averages of the model. These are normally approximate and found and checked empirically, and called “parametrizations". We have here a simple choice using two fitted coefficients:

$$\langle u'T'\rangle = -K_T \frac{\langle T\rangle}{\partial x},$$
$$\langle v'T'\rangle = -K_T \frac{\langle T\rangle}{\partial y},$$
$$\langle u'v'\rangle = -K_M \frac{\langle u\rangle}{\partial x}.$$

Acknowledgement: Section 4.5, “The Climate Modelling Primer", Fourth Edition, by Kendal McGuffie and Anne Henderson-Sellers, Wiley, 2014.

(9) Boundary Conditions

A fundamental assumption is that air and water do not enter land. Now this is generally false, but at the scale of geophysical flows we are considering both flows are negligible. These assumptions have several consequences for the boundary conditions.

(9.1) If the domain has a solid bottom:
Assume the equation of the bottom can be written in Cartesian coordinates $z = b(x, y)$ or $z - b = 0$ for all (x, y), where $b(x, y) \in \Re$ is a known differentiable function. Since there is zero flow into the bottom surface, the normal vector to the bottom surface and velocity vector must be orthogonal, i.e.

$$\left(\frac{\partial(z-b)}{\partial x}, \frac{\partial(z-b)}{\partial y}, \frac{\partial(z-b)}{\partial z}\right) \cdot (u, v, w) = 0.$$

Simplifying, we have the boundary condition when $z = b(x, y)$

$$w = u\frac{\partial b}{\partial x} + v\frac{\partial b}{\partial y}.$$

Therefore, for each point (x, y) we get a constraint on (u, v, w) so the flow is tangent to the bottom surface. Note that the derivative must be taken before the substitution $z - b = 0$.

(9.2) An ocean or sea free surface:
Here the sea moves with the fluid. For simplicity we ignore breaking waves, evaporation and precipitation, and assume we can describe the sea surface as a known explicit equation $z = \eta(x, y)$, or an unknown function to be determined by the model. Then since we are also assuming water does not enter or leave air, the surface of the sea must be a material surface, i.e. a parcel of water which is at the surface at one instant remains at the surface for the next instant. Hence we can write (recall d/dt is differentiation following the motion, the advective derivative) for $z = \eta(x, y)$

$$\frac{d}{dt}(z - \eta(x, y)) = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}(z - \eta(x, y)) = 0.$$

This simplifies to

$$w = \frac{\partial \eta}{\partial t} + u\frac{\partial \eta}{\partial x} + v\frac{\partial \eta}{\partial y}.$$

This is then the boundary condition at a free surface.

(9.3) The vertical boundary assumption for an ocean or sea:
In steps (1) and (2) we assumed $b(x, y)$ and $\eta(x, y)$ were known functions not dependent on time. Of course this is unrealistic, but numerical methods normally rely on the domain having need fixed boundaries. One simplification is to assume a vertical boundary, some way out from the foreshore or sea bed's actual moving boundary. Along this boundary the horizontal component of the flow is assumed to be zero. Given that $\eta(x, y)$ is an unknown function that must be solved for or eliminated from the model as an additional unknown, particular problems of a challenging nature can be involved in the solution. Adopting special forms for η interms of a modest number of unknown parameters is one possible approach.

(9.4) Sea surface pressure continuity:
We ignore surface tension which acts over too short a distance to be of any significance. Then the pressure at the sea-atmosphere interface must be continuous. Assume the sea pressure is hydrostatic, so its value at depth h with constant sea density ρ_0 is ρgh. If $z = 0$ at a reference sea level, and the sea has height function $z = \eta(x, y)$ with pressure a function of (x, y, z), then pressure continuity gives the pressure boundary condition

$$p_{sea}(x, y, 0) = p_{atm}(x, y, \eta(x, y)) + \rho_0 \eta(x, y) g.$$

(9.5) If viscosity is neglected:
Neglecting viscosity is a good assumption for sea water, and more generally when the height of the boundary layer is small compared with the scale under investigation. If so viscosity can be neglected entirely in the equations (equivalently set to zero) and the only constraint on the flow is its normal component along the boundary must be zero because of our preliminary assumptions.

(9.6) If viscosity is retained:
If a fluid is viscous then generally a "no slip" condition is applied at any fixed boundary, i.e. all fluid velocity components are zero. However, wind stress on the ocean is the most important source of upper ocean current flows, even though water has very low viscosity. Now the ocean surface is a moving

boundary. If τ^x and τ^y are the components of the wind stress caused by the atmosphere on the ocean then we get at the surface

$$\rho_0 \nu_E \frac{u}{z} = \tau^x, \text{ and } \rho_0 \nu_E \frac{v}{z} = \tau^y,$$

where ρ_0 is a reference sea water density, ν_E is the vertical eddy viscosity coefficient, and μ is the coefficient of dynamic viscosity.
The wind stresses are normally approximated by quadratic functions of the wind velocities, with zero values at the moving ocean surface and a value at height 10 m above the ocean surface, parameterized with a drag coefficient C_d, wind vector (u_{10}, v_{10}), wind speed $U_{10} = \sqrt{u_{10}^2 + v_{10}^2}$, so

$$\tau^x = C_d \rho_{air} U_{10} u_{10} \text{ and } \tau^y = C_d \rho_{air} U_{10} v_{10}.$$

Recall that the winds are driven by the differences in heating by the sun between the tropics and the polar regions.

(9.7) An open boundary at the edge of a model domain:
If a model domain is not of the whole earth but for a restricted region, then values of the unknown variables (and or their derivatives) at the edges of this region need to be specified. These could come from say satellite or other data, or a model covering a larger region. Generally, this aspect of modelling is very challenging, with many unknowns which must be estimated. For example coastal tidal modelling including esturine flows.

Acknowledgement: Section 4.6 "Introduction to geophysical fluid dynamics: physical and numerical aspects" 2nd edition, by Benoit Cushman-Roisin and Jean-Marie Beckers, AP, Elsevier, 2011.

(10) Relative Scales for Atmospheric and Ocean Flows

Typical scales

In the table below, L is the macroscopic horizontal length scale, H the vertical length scale, T the time scale, W the vertical component of flow velocity scale, P the pressure scale, and U the horizontal speed scale. There is no scale for temperature.

variables	scale	unit	atmosphere value	ocean value
x, y	L	m	$10^5\ m = 100\ km$	$10^4\ m = 10\ km$
z	H	m	$10^3\ m = 1\ km$	$10^2\ m = 100\ m$
t	T	s	$\geq 12\ hrs \approx 4 \times 10^4\ s$	$\geq 1\ day \approx 9 \times 10^4\ s$
u, v	U	m/s	$10\ m/s$	$0.1\ m/s = 3.6\ km/hr$

Table 1: Typical scales for some of the atmosphere and ocean variables.

We also have the variable scales W for w with units m/s, P for p with $kg/(ms^2)$ and $\Delta\rho$ for ρ with kg/m^3.

Scale analysis

(10.1) Fundamental assumed constaints:
The first two inequalities express the effect of rotation should be significant for the flow, and the third that effects which are considered should have a larger length scale than the height scale which is relatively very small:

$$\frac{1}{T} \lesssim \Omega,\ \frac{U}{L} \lesssim \Omega \text{ and } H \ll L.$$

(10.2) Continuity equation: Consider the simple form of the continuity equation

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0.$$

Taking the three terms in order they have scales

$$\frac{U}{L},\ \frac{U}{L},\ \frac{W}{H}.$$

If we had $U/L \ll W/H$ then the continuity equation would imply, in a first approximation, $\partial w/\partial z = 0$ so w would be constant in the vertical direction.

Considering either the bottom or top this would imply zero vertical velocity everywhere, which is impossible. Hence either or both $\partial u/\partial x$ or $\partial v/\partial y$ must be of significant size and then w will be small. We have either (i) $W/H \ll U/L$ and the flow is approximately two dimensional or (ii) $W/H \approx U/L$. In each of these cases we get using (1)

$$W \lesssim \frac{H}{L}U \implies W \ll U.$$

To summarize, large scale geophysical flows are shallow and approximately two dimensional, i.e. $H \ll L,\ W \ll U$.

(10.3) Horizontal momentum equations:

Scale consequences of the x-momentum and y-momentum equations are similar. Consider the scales if the 10 terms of the3 x-momentum equation in its turbulence averaged Boussinesq approximation form:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} + f_* w - fv = \\ -\frac{1}{\rho_0}\frac{\partial p}{\partial x} + \frac{\partial}{\partial x}\left(\mathscr{A}\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(\mathscr{A}\frac{\partial u}{\partial y}\right) + \frac{\partial}{\partial z}\left(\nu_E\frac{\partial u}{\partial x}\right).$$

Taking orders of magnitude, each term has in order the value:

$$\frac{U}{T},\ \frac{U^2}{L},\ \frac{U^2}{L},\ \frac{WU}{H},\ \Omega W,\ \Omega U,\ \frac{P}{\rho_o L},\ \frac{\mathscr{A}U}{L^2},\ \frac{\mathscr{A}U}{L^2},\ \frac{\nu_E U}{H^2}.$$

In what follows we derive consequences of the assumptions set out in (1).

(10.4) Vertical momentum equations:

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} - f_* u = \\ -\frac{1}{\rho_0}\frac{\partial p}{\partial z} - \frac{g\rho}{\rho_0} + \frac{\partial}{\partial x}\left(\mathscr{A}\frac{\partial w}{\partial x}\right) + \frac{\partial}{\partial y}\left(\mathscr{A}\frac{\partial w}{\partial y}\right) + \frac{\partial}{\partial z}\left(\nu_E\frac{\partial w}{\partial z}\right)$$

These 11 terms take the order of magnitude form in order

$$\frac{W}{T},\ \frac{UW}{L},\ \frac{W^2}{H},\ \frac{WU}{H},\ \Omega U,\ \Omega U,\ \frac{P}{\rho_o H},\frac{g\Delta\rho}{\rho_0},\ \frac{\mathscr{A}W}{L^2},\ \frac{\mathscr{A}W}{L^2},\frac{\nu_E W}{H^2}.$$

Following an analysis of the relative values of each term, we get the so-called hydrostatic balance approximate form

$$0 = -\frac{\partial p}{\partial z} - g\rho,$$

where we have replaced the pressure fluxuations $\Delta\rho$ or ρ' with ρ.

(11)Dimensionless numbers for geophysical flows:

Consider again the x-momentum equation in its simplified form

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} - fv = \\ -\frac{1}{\rho_0}\frac{\partial p}{\partial x} + \frac{\partial}{\partial x}\left(\mathscr{A}\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(\mathscr{A}\frac{\partial u}{\partial y}\right) + \frac{\partial}{\partial z}\left(\nu_E\frac{\partial u}{\partial z}\right).$$

The order of magnitude of the second and third terms is taken as U^2/L and of the seventh and eighth as $\mathscr{A}U/L^2$ to get a reduced set of eight ratios

$$\frac{U}{T},\ \frac{U^2}{L},\ \frac{WU}{H},\ \Omega U,\ \frac{P}{\rho_o L},\ \frac{\mathscr{A}U}{L^2},\ \frac{\nu_E U}{H^2}.$$

Dividing by the Coriolis ΩU term leads a sequence of dimensionless ratios

$$\frac{1}{\Omega T},\ \frac{U}{\Omega L},\ \frac{W}{\Omega H},\ 1,\ \frac{P}{\rho_o \Omega U L},\ \frac{\mathscr{A}}{\Omega L^2},\ \frac{\nu_E}{\Omega H^2}.$$

The 6 ratios give rise to 5 named and 1 unnamed dimensionless numbers which are used to describe bulk properties of geophysical flows. The named terms are the temporal Rossby number, the Rossby number, the Ekman number, the Reynolds number and the Richardson number. The significance of these numbers is described in what follows.

(11.1) Temporal Rossby number $\mathbf{Ro_T}$:

The definition

$$Ro_T := \frac{1}{\Omega T} = \frac{U/T}{\Omega U} = \frac{\text{time rate of change of horizontal velocity}}{\text{Coriolis force}} \lesssim 1.$$

(11.2) Rossby number Ro:

Using $U/L \lesssim \Omega$ from (1) we get

$$Ro := \frac{U}{\Omega L} = \frac{U^2/L}{\Omega U} = \frac{\text{advection}}{\text{Coriolos force}} \lesssim 1.$$

(11.3) First unnamed ratio $\mathbf{R_1}$:

$$R_1 := \frac{WL}{UH} \times Ro \lesssim Ro.$$

(11.4) Second unnamed ratio $\mathbf{R_2}$:

Generally for geophysical flows the pressure gradient scales along with the Coriolis force. This gives rise to

$$R_2 := \frac{P}{\rho_0 \Omega L U} \approx 1.$$

In other words, R_2 is not normally useful to describe these flows.

(11.5) Third unnamed ratio $\mathbf{R_3}$:

$$R_3 := \frac{\mathscr{A}}{\Omega L^2}$$

(11.6) Ekman number Ek:

$$Ek := \frac{\nu_E}{\Omega H^2} \lesssim 1.$$

The corresponding terms in the momentum equations must be retained because of the important Ekman boundary layer effect created by vertical friction.

(11.7) Reynolds number Re:

A commonly occurring named dimensionless number in engineering fluid dynamics is the Reynolds number. It measures the ratio of the inertial and frictional forces. For geophysical flows it is very large. We have

$$Re := \frac{UL}{\nu_E} = \frac{U}{\Omega L} \times \left(\frac{L}{H}\right)^2 = \frac{Ro}{Ek}\left(\frac{L}{H}\right)^2 \lesssim \frac{Ro}{Ek}.$$

(11.8) Richardson number Ri:

This dimensionless quantity is important for geophysical flows since it gives an indication of the significance of stratification for any particular flow in any particular region. To derive it we make two assumptions, namely that the pressure gradient is balanced by the gravitational force as in the Bossinesque approximation z-momentum simplified equation, and that it scales according to the Coriolis term in the momentum equations. Thus we assume

$$\frac{P}{H} \approx g\Delta\rho \text{ and } \frac{\rho_0 P}{L} \approx \Omega U.$$

Then the Richardson number times **Ri** is defined as the ratio of the left to the right hand side of the first approximation. It can have values with orders less than 1 when stratification effects are negligible, about 1 when they are significant, and much greater than 1 when they are dominant for the flow. In detail using the two equations given above

$$\frac{g\Delta\rho}{P/H} = \frac{gH\Delta\rho}{\rho_0 U^2} = \frac{U}{\Omega L} \times \frac{gH\Delta\rho}{\rho_0 U^2} = Ro \times R_i,$$

so

$$Ri = \frac{gH\Delta\rho}{\rho_0 U^2}.$$

Acknowledgement: Sections 1.4, 4.3, “Introduction to geophysical fluid dynamics: physical and numerical aspects" 2nd edition, by Benoit Cushman-Roisin and Jean-Marie Beckers, AP, Elsevier, 2011.

(12) Planetary waves

Standard definitions for wave motion

Consider a two dimensional wave of **amplitude** A and phase α where

$$\alpha = k_x x + k_y y - \omega t + \varphi.$$

so the "signal" has the form

$$a = A\cos(k_x x + k_y y - \omega t + \varphi)$$

The extension to three space dimensions is straight forward.
The real number ω is the (angular) **frequency** and φ the so called **reference phase**. The **wave number vector k** Has the form $\mathbf{k} = (k_x, k_y)$ with magnitude $k = \sqrt{k_x^2 + k_y^2}$, called the **wavenumber** of the wave.
Fix an instant t and consider the shape of the wave as a graph over the x, y-plane. We call **crests** the highest points with value A and **troughs** the lowest with value $-A$. The **wave length** λ is defined to be the distance between two closest crests. If we define

$$\lambda_x := \frac{2\pi}{k_x} \text{ and } \lambda_y := \frac{2\pi}{k_y},$$

then some simple trigonometry shows

$$\lambda^2 = \frac{4\pi^2}{k_x^2 + k_y^2} \quad \Longrightarrow \quad \lambda = \frac{2\pi}{k}$$

If we fix a point in space (x, y) and let time vary, then the shortest time between two maximum signals passing this point is called the **period** of the wave and has the value

$$T = \frac{2\pi}{\omega}.$$

The propagation speed in the x-direction is $c_x := \omega/kx$ and in the y-direction $c_y := \omega/k_y$. The actual propagation of the wave should be measured perpendicular to the wave crest as it passes (x, y) and this, using a similar calculation to that given for λ above, can be shown to be

$$c = \frac{\omega}{k}.$$

We have assumed that the wave numbers and frequency are independent. However, in the case where there is **dispersion**, a relation will exist so that ω is a function of k, which we write $\omega(k_x, k_y)$. The actual form of the relation is called naturally a **dispersion relation**. It depends on the physical properties of the medium which supports the wave.
In the case where a wave is the supposition of two waves of similar amplitude and close wave numbers then they could cancel then their signs are opposite and reinforce when their signs are the same. For example if the amplitudes are equal and the waves are in one space dimension we could write for the composite signal using a trigonometric identity

$$a = A\cos(k_1 x - \omega_1 t) + A(\cos(k_2 x - \omega_2 t) = 2A\cos\left(\tfrac{1}{2}\Delta k x - \tfrac{1}{2}\Delta\omega t\right)\cos(kx - \omega t),$$

where $\Delta\omega := \omega_1 - \omega_2$, $\Delta k := k_1 - k_2$, $k := (k_1 + k_2)/2$ and $\omega := (\omega_1 + \omega_2)/2$. Let us also assume the two wave numbers are very close so $\Delta k \ll k_1 - k_2$ and thus given a continuous dispersion relation $\Delta\omega \ll \omega_1 - \omega_2$. Then the first cosine represents a wave with small wave number and thus long wavelength and similarly low frequency. The second consine however has parameters close to the originals. Since the two cosines are multiplied the wave shape is modified giving an amplitude

$$2A\cos\left(\tfrac{1}{2}\Delta k x - \tfrac{1}{2}\Delta\omega t\right)$$

which is a wave with speed $\Delta\omega/\Delta k$, an approximation to the so-called **group velocity**

$$c_g := \frac{d\omega}{dk}.$$

Rossby waves

Consider a geophysical fluid which is homogeneous and invicid, satisfies the shallow water model and the scales

$$Ro = \frac{U}{\Omega L} \ll 1 \text{ and } R_T = \frac{1}{\Omega T} \ll 1 \quad \Longrightarrow \quad \frac{L}{T} \gg U,$$

which describe slow flow fields (small U) which evolve fast. Then we can write the horizontal momentum equations, using the depth $\eta = h - H$, where H is the mean depth, and Coriolis parameter $f = 2\Omega\sin\varphi$, as

$$\frac{\partial u}{\partial t} - fv = -g\frac{\partial \eta}{\partial x},$$

$$\frac{\partial v}{\partial t} + fu = -g\frac{\partial \eta}{\partial y},$$

and note that such a flow is vertically homogeneous so the independent variables are x, y, t.
We also assume the bottom is flat so we can regard H as a constant and write the continuity equation from the pages on Geostrophic and Barotropic Flows.

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(hu) + \frac{\partial}{\partial y}(hv) = 0$$

as

$$\frac{\partial \eta}{\partial t} + \left(u\frac{\partial \eta}{\partial x} + v\frac{\partial \eta}{\partial y}\right) + H\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + \eta\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0.$$

Next we apply a scale analysis to this set of four terms. In order we have, using $\Delta\eta$ as a scale for the sea-level height variation,

$$\frac{\Delta\eta}{T},\ \frac{U\Delta\eta}{L},\ \frac{HU}{L},\ \frac{\Delta\eta U}{L}.$$

We can consequently ignore the second and fourth terms in the full continuity equation, leading to a simplified linear form of the equation to complete a 3×3 system with

$$\frac{\partial \eta}{\partial t} + H\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0.$$

This requires $\Delta\eta \ll H$ so waves must be of relatively small amplitude.

Consider the Coriolis parameter where the latitude φ is not necessarily constant:

$$f = 2\Omega \sin\varphi.$$

If we expand in a Taylor series using $\varphi = \varphi_0 + y/a$, where $a = 6371$ is the earth radius and y is small, $f_0 := 2\Omega \sin\varphi_0$, $\beta_0 := 2(\Omega/a)\cos\varphi_0$ (the **beta-parameter**) and retaining only the first two series terms (effectively linearizing about $\varphi = \varphi_0$), we can write

$$f = 2\Omega\sin\varphi_0 + (\frac{2\Omega}{a}\cos\varphi_0)y + \cdots \approx f_0 + \beta_0 y.$$

Typical mid-latitude values for the constants are $f_0 = 8 \times 10^{-5}s^{-1}$ and $\beta_0 = 2 \times 10^{-11}m^{-1}s^{-1}$. If β_0 is set to zero, then we have what's called the f-plane.

If retained then we refer to the flow as being in the β-plane. This rather quaint jargon is common in the field. To add to this list, we also have the dimensionless **planetary number**:

$$\beta := \frac{\beta_0 L}{f_0} \ll 1.$$

In this situation we can write a 3×3 set of model equations

$$\begin{aligned}
\frac{\partial u}{\partial t} - (f_0 + \beta_0 y)v &= -g\frac{\partial \eta}{\partial x}, \\
\frac{\partial v}{\partial t} + (f_0 + \beta_0 y)u &= -g\frac{\partial \eta}{\partial y}, \\
\frac{\partial \eta}{\partial t} + H\left(\frac{\partial \eta}{\partial x} + \frac{\partial \eta}{\partial y}\right) &= 0.
\end{aligned}$$

Here, the large terms are f_0, g, H. In f-plane with geostrophic dynamics (i.e. invicid steady flow) we get in a first approximation, values for u, v which we can substitute back in the model equations and solve again for u, v to get the expressions

$$\begin{aligned}
u &= -\frac{g}{f_0}\frac{\partial \eta}{\partial y} - \frac{g}{f_0^2}\frac{\partial^2 \eta}{\partial x \partial t} + \frac{\beta_0 g}{f_0^2} y \frac{\partial \eta}{\partial y}, \\
v &= +\frac{g}{f_0}\frac{\partial \eta}{\partial x} - \frac{g}{f_0^2}\frac{\partial^2 \eta}{\partial y \partial t} + \frac{\beta_0 g}{f_0^2} y \frac{\partial \eta}{\partial x}.
\end{aligned}$$

Substituting the space derivatives back in the model continuity equation then gives

$$\frac{\partial \eta}{\partial t} - R^2 \frac{\partial}{\partial t}\nabla^2 \eta - \beta_0 R^2 \frac{\partial \eta}{\partial x} = 0,$$

where $R := \sqrt{gH}/f_0$ is the so-called **deformation radius**.
For a solution consisting of a single Fourier mode $\eta = \eta(t, x, y) = \cos(k_x x + k_y y - \omega t)$ we get the dispersion relation

$$\omega = -\frac{\beta_0 R^2 k_x}{1 + R^2(k_x^2 + k_y^2)}.$$

which gives the frequency as a function of the wave number components. If $\beta_0 = 0$ then we get steady geostrophic flow in the f-plane.

Consequences of the dispersion relation for planetary waves

(12.1) The wave speed $c_x = \omega/k_x$ is always negative. This implies the wave propagates to the west, i.e. only in a NW, W or SW direction.

(12.2) Very long waves (those with $1/k_x \gg R$ and $1/k_y \gg R$ always travel west with a speed $-\beta_0 R^2$, the maximum speed under these assumptions.

(12.3) Planetary waves have a maximum frequency: to see this consider the contours of constant frequency ω in the (k_x, k_y) plane. They consist of circles with equation

$$\left(k_x + \frac{\beta_0}{\omega}\right)^2 + k_y^2 = \left(\frac{\beta_0^2}{4\omega^2} - \frac{1}{R^2}\right) = \text{square of the radius.}$$

To obtain a real circle we must have therefore $\beta_0^2 > 4\omega^2/R^2$. This implies there is a maxium frequency for planetry waves, namely

$$|\omega|_{\max} = \beta_0 R/2.$$

(12.4) The group velocity, namely $(\frac{\partial\omega}{\partial k_x}, \frac{\partial\omega}{\partial k_y})$ points inwards to the center of the circles of constant ω in the (k_x, k_y) plane. This shows that long waves have westward group velocities, but energy is carried eastwards by the shorter waves.

(13) Geostrophic and barotropic flows

Consider flows where the Coriolis force exceeds all other acceleration terms on the left of the flow equations, i.e. rotational dominates. Assume the fluid is homogeneous and all frictional terms can be set to zero. We call such flows **geostrophic**. We can describe this situation with the estimates

$$Ro_T \ll 1,\ Ro \ll 1, \quad \text{and } Ek \ll 1,$$

where recall these dimensionless quantities are defined by

$$Ro_T := \frac{1}{\Omega T},\ Ro := \frac{U}{\Omega L},\ \text{and } E_k := \frac{\nu_E}{\rho_0 \Omega L U}.$$

This results in a simplified set of equations for u, v, w, p with the Coriolis parameter $f = \Omega \cos \varphi$ and φ the latitude in radians:

$$\begin{aligned} -fv &= -\frac{1}{\rho_0}\frac{\partial p}{\partial x}, \\ +fu - \frac{1}{\rho_0}\frac{\partial p}{\partial y}, \\ 0 &= -\frac{1}{\rho_0}\frac{\partial p}{\partial z}, \\ 0 &= \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}. \end{aligned}$$

Simple manipulations of these equations results in the three relations

$$\frac{\partial u}{\partial z} = 0,\ \frac{\partial v}{\partial z} = 0,\ \text{and } (u, v) \cdot (\frac{\partial p}{\partial x}, \frac{\partial p}{\partial y}) = 0,$$

which hold through all points in the domain where the geostrophic assumptions are valid.

From the first two relations we see that all particles on the same vertical line move together, i.e. the fluid is vertically rigid. This is simply false for cyclones and anticyclones for example.

From the second relation we see that the fluid flow is perpendicular to the pressure gradient so the lines of constant pressure (the isobars) are streamlines. Now this is approximately true for cyclones and anticyclones.

If the flow extends over a region which is not too latitudinally wide so the Coriolis parameter can be taken as constant, then we have another consequence of the simplified equations, namely

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \implies \frac{\partial w}{\partial z} = 0.$$

Thus the vertical velocity is independent of height. If for example the flow bottom is flat so $w = 0$ there, we must have $w = 0$ throughout every vertical column above a flat point, and the flow is then strictly 2D.

Geostrophic flows over a bottom described with a given function

Let the bottom height be given by a function $b = b(x, y)$. From the section on boundary conditions we derived the relationship which holds at each point $(x, y, b(x, y)$ on the bottom surface

$$w = u\frac{\partial b}{\partial x} + v\frac{\partial b}{\partial y}.$$

But, as seen before, the vertical component of velocity is independent of z. Since it is zero at the top of each vertical column we must have $w = 0$ throuout the column and at the bottom

$$u\frac{\partial b}{\partial x} + v\frac{\partial b}{\partial y} = 0$$

so the flow cannot climb up or down the bottom slope. This fact, plus vertical rigidity implies the flow follows the closed contours of $b(x, y) = constant$ (called isobaths). Hence, since the flow is geostrophic and the flow is along the pressure contours (isobars), for bounded regions where a contour meets the boundary, the velocity must be zero. Thus there is no flow at all along these contours.

(14) Barotropic flows and the shallow water model

Now continue to assume that the fluid is homogeneous with zero viscosity, but the Coriolis acceleration is not so large that time derivatives can be neglected. The equations are

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} - fv = -\frac{1}{\rho_0}\frac{\partial p}{\partial x},$$

$$
\begin{aligned}
\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} + fu - \frac{1}{\rho_0}\frac{\partial p}{\partial y}&, \\
0 &= -\frac{1}{\rho_0}\frac{\partial p}{\partial z}, \\
0 &= \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}.
\end{aligned}
$$

An examination of the form of the first three equations shows that if the horizontal flow field (u, v) is initially independent of depth z, then it will continue to be independent for all applicable time. We call such flows without (u, v) vertical variation **barotropic**. The equations can be simplified to

$$
\begin{aligned}
\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} - fv &= -\frac{1}{\rho_0}\frac{\partial p}{\partial x}, \\
\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + fu - \frac{1}{\rho_0}\frac{\partial p}{\partial y}&, \\
0 &= -\frac{1}{\rho_0}\frac{\partial p}{\partial z}, \\
0 &= \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}.
\end{aligned}
$$

Let $h = h(t, x, y)$ be the depth of the fluid column above the bottom surface and integrate the continuity equation with respect to z, using the property that

$$
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}
$$

is z-independent, to get

$$
\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)\int_b^{b+h} 1\ dz + w|_b^{b+h} = 0.
$$

Usng the property that surface particles cannot leave the surface and bottom particles cannot go through the bottom ((4.28) and (4.31) check), setting as before $\eta := b + h - H$ where $H > 0$ it a suitable fluid height reference level, recalling $w = w(z)$ we get

$$
\begin{aligned}
w(b+h) &= \frac{\partial}{\partial t}(b+h) + u\frac{\partial}{\partial x}(b+h) + v\frac{\partial}{\partial y}(b+h) \\
&= \frac{\partial \eta}{\partial t} + u\frac{\partial \eta}{\partial x} + v\frac{\partial \eta}{\partial y},
\end{aligned}
$$

$$w(b) = u\frac{\partial b}{\partial x} + v\frac{\partial b}{\partial y}.$$

These relations enable us to simplify the equation we obtained integrating with respect to z to get

$$\frac{\partial \eta}{\partial t} + u\frac{\partial}{\partial x}(hu) + v\frac{\partial}{\partial y}(hv) = 0.$$

Finally, since atmospheric pressure is constant across the fluid surface, and the fluid is homogeneous, we assume the dynamic pressure (called previously p') satisfies

$$p = \rho_0 g \eta$$

as its constant value independent of z. Using this expression we obtain three dependent variables (u, v, η) and three independent variables (t, x, y) giving the so-called **shallow water model**

$$\begin{aligned}
\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} - fv &= -g\frac{\partial \eta}{\partial x}, \\
\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + fu &= -g\frac{\partial \eta}{\partial y}, \\
\frac{\partial \eta}{\partial t} + \frac{\partial}{\partial x}(hu) + \frac{\partial}{\partial y}(hv) &= 0.
\end{aligned}$$

If the bottom is flat (so we set $b \equiv 0$ or $\eta = h$), this simplifies to

$$\begin{aligned}
\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} - fv &= -g\frac{\partial h}{\partial x}, \\
\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + fu &= -g\frac{\partial h}{\partial y}, \\
\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(hu) + \frac{\partial}{\partial y}(hv) &= 0.
\end{aligned}$$

Vorticity dynamics

In this section we derive a conservation law for friction free (invicid) barotropic flows. If we subtract the y-derivative of the first equation from the x-derivative of the second equation and use the material derivative

$$\frac{d}{dt} := \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y},$$

with quite a lot of careful manipulation and noting $\frac{\partial f}{\partial t} = 0$ since $f := \Omega \sin \varphi$ where φ is latitude, we get the expression

$$\frac{d}{dt}\left(f + \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) + \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)\left(f + \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = 0.$$

Noting that the **vorticity** of a two dimensional flow is defined as $\zeta := \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$, we can then write

$$\frac{d}{dt}(f + \zeta) + \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)(f + \zeta) = 0.$$

Next, using the form of the barotropic continuity equation with $\eta = h$, the depth of the fluid parcel, we can write the form as

$$\frac{dh}{dt} + \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) h = 0.$$

Note that the material derivative is a derivation - i.e. it is linear over constants and satisfies the derivative product rule and the quotient rule. (But not too much more, for example Taylor's theorem!). The latter property and the two equations we have derived can be used to show easily

$$\frac{d}{dt}\left(\frac{f + \zeta}{h}\right) = 0.$$

Thus, a fluid parcel moving with the flow conserves the value of quotient $(f + \zeta)/h$.

Acknowledgement: Sections 7.3-7.4, “Introduction to geophysical fluid dynamics: physical and numerical aspects" 2nd edition, by Benoit Cushman-Roisin and Jean-Marie Beckers, AP, Elsevier, 2011.

(15) Quasi-geostrophic flows

The equations for a quasi-geostrophic flow apply when in the horizontal momentum equations, the terms representing relative acceleration, advection and friction or viscosity are all negligible. There are great advantages mathematically in studying geostrophic flows because the time derivatives and advection are absent. Thus flows which are close to geostrophic are used a great deal in geophysical modelling. Here we give a single partial differential equation for the pressure residual for a stratified flow using a stream function. In terms of this variable, all the components of velocity, together with the pressure and density residuals can be evaluated.

In the numerical link there are given several methods for computing the Jacobian $J(a,b)$ which appears frequently in the equations.

We assume $\overline{\rho}$ and $\overline{p}$ are given functions of height z which satisfy hydrostatic equlibrium:

$$\frac{\partial \overline{p}}{\partial z} = -\overline{\rho} g.$$

Assume also the the actual density and pressure satisfy perturbation relations

$$\begin{aligned} \rho &= \overline{\rho}(z) + \rho'(x,y,z,t), \quad \text{such that } |\rho'| \ll |\overline{\rho}|, \\ p &= \overline{p}(z) + p'(x,y,z,t), \quad \text{such that } |p'| \ll |\overline{p}|. \end{aligned}$$

The so-called stratification frequency is defined by

$$N^2(z) := -\frac{g}{\rho_0}\frac{d\overline{\rho}}{dz}.$$

Jacobian determinant for dimension 2 is:

$$J(a,b) := \frac{\partial a}{\partial x}\frac{\partial b}{\partial y} - \frac{\partial a}{\partial y}\frac{\partial b}{\partial x}.$$

We also write ∇^2 for the horizontal Laplacian:

$$\nabla^2 \psi := \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2}.$$

Derivation of the quasi-geostrophic equation:

In this section we derive in a number of steps the so-called quasi-geostrophic equation for the time derivative of $q = q(x,y,z,t)$ the potential vorticity and $\psi = \psi(x,y,z,t)$ a stream function for p', both defined below.

(1) First we have a form of the flow equations, assuming we can assume we have an absence of friction and diffusion and are on the beta plane, and using $|\rho'| \ll \overline{\rho}$ in the density equation and stratification in the pressure equation:

$$\begin{aligned} \frac{du}{dt} - f_0 v - \beta_0 y v &= -\frac{1}{\rho_0}\frac{\partial p'}{\partial x}, \\ \frac{dv}{dt} + f_0 u + \beta_0 y u &= -\frac{1}{\rho_0}\frac{\partial p'}{\partial y}, \\ \frac{\partial p'}{\partial z} + \rho' g &= 0, \\ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} &= 0, \\ \frac{\partial \rho'}{\partial t} + u\frac{\partial \rho'}{\partial x} + v\frac{\partial \rho'}{\partial y} + w\frac{d\overline{\rho}}{dz} &= 0. \end{aligned}$$

(2) Next we consider the expressions for $u = u_g, v = v_g$ in geostrophic equilibrium which we can write:

$$-f_0 v_g = -\frac{1}{\rho_0}\frac{\partial p'}{\partial x} \text{ and } + f_0 u_g = -\frac{1}{\rho_0}\frac{\partial p'}{\partial y}.$$

We allow for a small deviation (u', v') from geostrophic equilibrium and write

$$u = u_g + u' \text{ and } v = v_g + v'.$$

In the small time derivative and advective terms of the x, y momentum equations we use the geostrophic values (u_g, v_g), but in the Coriolis terms use $(u_g + u', v_g + v')$ and neglect vertical advection. Using the Jacobian determinant J we get for the (x, y) momentum equations the forms

$$\begin{aligned} -\frac{1}{\rho_0 f_0}\frac{\partial^2 p'}{\partial y \partial t} - \frac{1}{\rho_0^2 f_0^2} J\left(p', \frac{\partial p'}{\partial y}\right) - f_0 v - \frac{\beta_0}{\rho_0 f_0} y \frac{\partial p'}{\partial x} &= -\frac{1}{\rho_0}\frac{\partial p'}{\partial x}, \\ +\frac{1}{\rho_0 f_0}\frac{\partial^2 p'}{\partial x \partial t} + \frac{1}{\rho_0^2 f_0^2} J\left(p', \frac{\partial p'}{\partial x}\right) + f_0 u - \frac{\beta_0}{\rho_0 f_0} y \frac{\partial p'}{\partial y} &= -\frac{1}{\rho_0}\frac{\partial p'}{\partial y}, \end{aligned}$$

We solve these two equations for u, v and obtain corrections to the geostrophic approximation terms, namely

$$u = -\frac{1}{\rho_0 f_0}\frac{\partial p'}{\partial y} - \frac{1}{\rho_0 f_0^2}\frac{\partial^2 p'}{\partial x \partial t} - \frac{1}{\rho_0^2 f_0^3} J\left(p', \frac{\partial p'}{\partial x}\right) + \frac{\beta_0}{\rho_0 f_0^2} y \frac{\partial p'}{\partial y},$$

$$
\begin{aligned}
&= u_g + u', \\
v &= +\frac{1}{\rho_0 f_0}\frac{\partial p'}{\partial x} - \frac{1}{\rho_0 f_0^2}\frac{\partial^2 p'}{\partial y \partial t} - \frac{1}{\rho_0^2 f_0^3} J\left(p', \frac{\partial p'}{\partial y}\right) - \frac{\beta_0}{\rho_0 f_0^2} y \frac{\partial p'}{\partial x}. \\
&= v_g + v'.
\end{aligned}
$$

(3) We substitute the x and y derivatives respectively of these expressions in the full continuity equation and solve for the z derivative of w to obtain

$$
\frac{\partial w}{\partial z} = \frac{1}{\rho_0 f_0^2}\frac{\partial}{\partial t}\nabla^2 p' + \frac{1}{\rho_0^2 f_0^3} J(p', \nabla^2 p') + \frac{\beta_0}{\rho_0 f_0^2}\frac{\partial p'}{\partial x}.
$$

(4) Considering the density equation, we observe the middle terms are both negligible compared with the other two terms so can derive the form

$$
\frac{\partial \rho'}{\partial t} + u\frac{\partial \rho'}{\partial x} + v\frac{\partial \rho'}{\partial y} - \frac{\rho_0 N^2}{g} w = \frac{\partial \rho'}{\partial t} + \frac{1}{\rho_0 f_0} J(p', \rho') - \frac{\rho_0 N^2}{g} w = 0.
$$

Dividing by N^2/g, differentiating with respect to z and eliminating density using the equation

$$
\frac{\partial p'}{\partial z} + \rho' g = 0,
$$

we get

$$
\frac{\partial}{\partial t}\left(\frac{\partial}{\partial z}\left(\frac{1}{N^2}\frac{\partial p'}{\partial z}\right)\right) + \frac{1}{\rho_0 f_0} J\left(p', \frac{\partial}{\partial z}\left(\frac{1}{N^2}\frac{\partial p'}{\partial z}\right)\right) + \rho_0 \frac{\partial w}{\partial z} = 0.
$$

(5) Next we take the equations derived in Steps (3) and (4) and eliminate $\partial w/\partial z$ to get an equation in p'. This is the quasi-geostrophic equation for motions in a continuously stratified fluid on the beta plane:

$$
\begin{aligned}
&\frac{\partial}{\partial t}\left(\nabla^2 p' + \frac{\partial}{\partial z}\left(\frac{f_0^2}{N^2}\frac{\partial p'}{\partial z}\right)\right) \\
&+ \frac{1}{\rho_0 f_0} J\left(p', \nabla^2 p' + \frac{\partial}{\partial z}\left(\frac{f_0^2}{N^2}\frac{\partial p'}{\partial z}\right)\right) - \beta_0 \frac{\partial p'}{\partial x} = 0.
\end{aligned}
$$

(6) Next we simplify the equation from Step (5) by introducing two new functions, a stream function for p' denoted ψ and the so-called potential vorticity q defined respectively by

$$
\psi := \frac{p'}{\rho_0 f_0},
$$

$$q := \nabla^2\psi + \frac{\partial}{\partial z}\left(\frac{f_0^2}{N^2}\frac{\partial \psi}{\partial z}\right) + \beta_0 y.$$

Using ψ and q we obtain the potential vorticity and stream function form of the quasi-geostrophic equation:

$$\frac{\partial q}{\partial t} + J(\psi, q) = 0.$$

This completes the derivation.

Given values for ψ and q we can find the corresponding values for flow variables, pressure and density residuals via

$$-f_0 v_g = -\frac{1}{\rho_0}\frac{\partial p'}{\partial x}, \ +f_0 u_g = -\frac{1}{\rho_0}\frac{\partial p'}{\partial y},$$

and

$$\frac{\partial \rho'}{\partial t} + \frac{1}{\rho_0 f_0}J(p', \rho') - \frac{\rho_0 N^2}{g}w = 0,$$

We get

$$\begin{aligned}
u_g &= -\frac{\partial \psi}{\partial y},\\
v_g &= +\frac{\partial \psi}{\partial x},\\
u' &= -\frac{1}{f_0}\frac{\partial^2 \psi}{\partial t \partial x} - \frac{1}{f_0}J\left(\psi, \frac{\partial \psi}{\partial x}\right) + \frac{\beta_0}{f_0}y\frac{\partial \psi}{\partial y},\\
v' &= -\frac{1}{f_0}\frac{\partial^2 \psi}{\partial t \partial y} - \frac{1}{f_0}J\left(\psi, \frac{\partial \psi}{\partial y}\right) + \frac{\beta_0}{f_0}y\frac{\partial \psi}{\partial x},\\
w &= -\frac{f_0}{N^2}\frac{\partial^2 \psi}{\partial t \partial z} + \frac{f_0}{N^2}J\left(\psi, \frac{\partial \psi}{\partial z}\right),\\
p' &= \rho_0 f_0 \psi,\\
\rho' &= -\frac{\rho_0 f_0}{g}\frac{\partial \psi}{\partial z}.
\end{aligned}$$

These might be easiest to derive in the order 6th, 7th, 1st, 2nd, 3rd, 4th, 5th.

Acknowledgement: Chapter 16, "Introduction to geophysical fluid dynamics: physical and numerical aspects" 2nd edition, by Benoit Cushman-Roisin and Jean-Marie Beckers, AP, Elsevier, 2011.

See also: "Gephysical fluid dynamics", 2nd edition, by J. Pedlosky, Springer-Verlag, 1987.