General Möbius Inversion

Recall $\sum_{d \mid n} \mu(d) = \begin{cases} 1 & n = 1 \\ 0 & n > 1 \end{cases}$

(Thm 4.8(ii))

Lemm. $F(x) = \sum_{n \le x} g\left(\frac{x}{n}\right) \Rightarrow$

$$g(x) = \sum_{n \le x} \mu(n) F\left(\frac{x}{n}\right)$$

Proof $g(x) = \sum_{n \le x} \Big(\underbrace{\sum_{d \mid n} \mu(d)}_{\substack{1 \Leftrightarrow n=1 \\ \text{else } 0}} \Big) g\left(\frac{x}{n}\right).$

$$= \sum_{dm \le x} \mu(d)\, g\left(\frac{x}{dm}\right) \qquad n = dm$$

$$= \sum_{d \le x} \mu(d) \Big(\sum_{m \le \frac{x}{d}} g\left(\frac{x}{dm}\right) \Big)$$

$$= \sum_{d \le x} \mu(d)\, F\left(\frac{x}{d}\right) \quad //$$

Similarly

Lemm (Th 4.8 (iv)) $F(x) = \sum_{n \le x} f(n) \Rightarrow$

$$\sum_{m \le x} F\left(\frac{x}{m}\right) = \sum_{d \le x} f(d) \left\lfloor \frac{x}{d} \right\rfloor = \sum_{n \le x} \sum_{d \mid n} f(d).$$

Application

$$\sum_{n\le x}\frac{\mu(n)}{n} = O(1) \quad \text{(i.e. the partial sums are bdd)}$$

Let $M(x) = \sum_{n\le x}\mu(n)$. By Thm 4.8 (iv) using $M = F$

$$\sum_{m\le x} M\left(\frac{x}{m}\right) = \sum_{d\le x}\mu(d)\left\lfloor\frac{x}{d}\right\rfloor = \sum_{n\le x}\sum_{d\mid n}\mu(d) = 1$$

Hence $$\sum_{d\le x}\mu(d)\left(\frac{x}{d} - \left\{\frac{x}{d}\right\}\right) = 1$$

$$\Rightarrow \quad x\sum_{d\le x}\frac{\mu(d)}{d} + O(x) = 1 \Rightarrow \sum_{d\le x}\frac{\mu(d)}{d} = O(1)$$

Thm 5.1 [D.J. Newman, 1980] If $a_n\in\mathbb{C}$, $|a_n|\le 1$, and $\sum_{n=1}^{\infty}\frac{a_n}{n^s} = F(s)$ converges to an analytic function on $\sigma>1$, and $F(s)$ can be extended to an analytic function on an open superset $\Omega\supset\{s : \mathrm{Re}(s)\ge 1\}$. Then the series converges at all $s = 1+it$, $\forall t\in\mathbb{R}$.

Proof (Later)

Application
Thm 5.2 $$\sum_{d\le x}\frac{\mu(d)}{d} = 0 \longrightarrow$$

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In[3102]:= ListLinePlot[Table[Sum[MoebiusMu[n] / n, {n, 1, m}], {m, 1, 1000}],
        ImageSize → 500, AxesLabel → {"n", ""}, AxesStyle → Thick]
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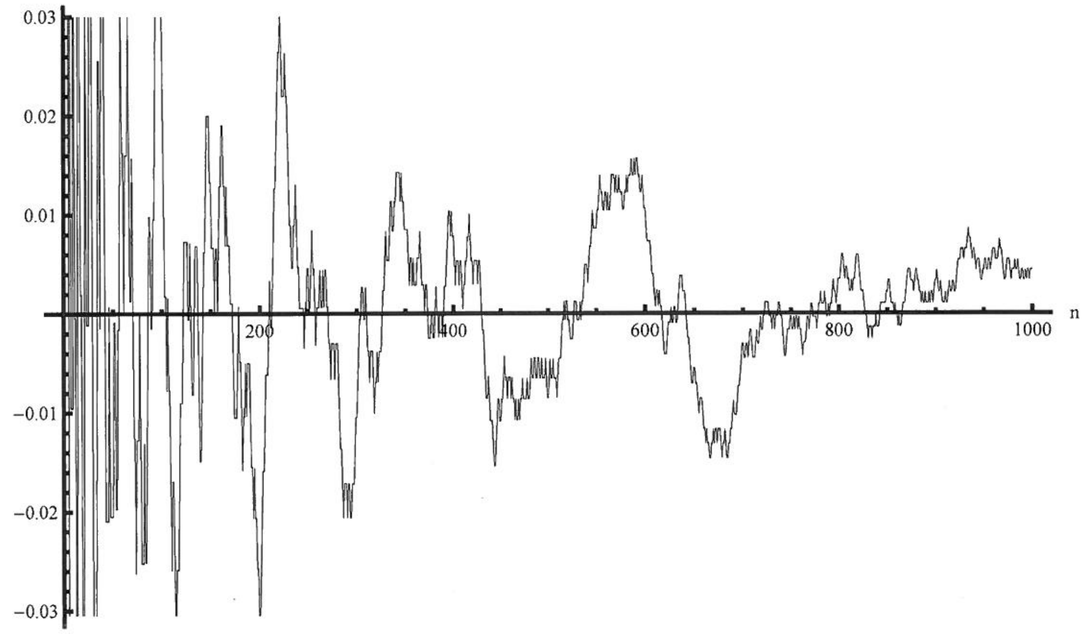
Out[3102]=



Theorem 5.2 $\displaystyle\sum_{n=1}^{\infty} \frac{\mu(n)}{n} = 0$

Proof $\sigma > 1$ $\quad \zeta(s) = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1} \Rightarrow \frac{1}{\zeta(s)} = \prod_p \left(1 - \frac{1}{p^s}\right)$

$\neq 0$

Recall the defn of $\mu(n)$ and multiply out the RHS of the product

$$\begin{aligned} &\left(1 - \tfrac{1}{2^s}\right) \\ &\times\left(1 - \tfrac{1}{3^s}\right) \\ &\times\left(1 - \tfrac{1}{5^s}\right) \\ &\vdots \end{aligned} \quad = 1 - \frac{1}{2^s} - \frac{1}{3^s} + \frac{0}{4^s} - \frac{1}{5^s} + \frac{(-1)^2}{2^s \cdot 3^s} + \dots + \frac{(-1)^r}{p_1^s \cdots p_r^s} + \dots$$

$$= \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} \quad \boxtimes$$

But for $\sigma > 0$ $\quad \zeta(s) = \frac{s}{s-1} - s\int_0^\infty \frac{\{x\}}{x^{s+1}}\,dx$ so

$(s-1)\zeta(s) = s - s(s-1)\int_0^\infty \frac{\{x\}}{x^{s+1}}\,dx$ is analytic in $\sigma > 0$

& has no zeros in $\sigma \geq 1$ (RHS $\neq 0$ at $s = 1$ since RHS $= 1$).

Hence, by Thm 5.1, the series $\boxtimes$ converges at $s = 1$ to $\displaystyle\sum_{n=1}^{\infty} \frac{\mu(n)}{n}$

But for $s \neq 1$ $\quad \dfrac{1}{\zeta(s)} = \dfrac{1}{\frac{s}{s-1} - s\int_0^\infty \cdots} = \dfrac{s-1}{s - s(s-1)\int_0^\infty \cdots} \xrightarrow[s \to 1]{} 0 \Rightarrow \boxtimes = 0$

Thm 5.4 [PNT] As $x \to \infty$ $\pi(x) \sim \frac{x}{\log x}$

Proof: Let $$F(x) := \sum_{n \le x} \left(\psi\left(\frac{x}{n}\right) - \left\lfloor \frac{x}{n} \right\rfloor + 2\gamma \right).$$

where $$\psi(x) = \sum_{p^k \le x} \log p = \sum_{n \le x} \Lambda(n)$$ (ch 4 start)

Apply Möbius inversion to get. (Thm 4.8 ~ Ex) — (1)
$$\psi(x) - \lfloor x \rfloor + 2\gamma = \sum_{n \le x} \mu(n) F\left(\frac{x}{n}\right)$$

Claim. $$\frac{\left(\sum_{n \le x} \mu(n) F\left(\frac{x}{n}\right) \right)}{x} \to 0 \text{ as } x \to \infty.$$ — (2)

now $$F(x) = \sum_{n \le x} \psi\left(\frac{x}{n}\right) - \sum_{n \le x} \left\lfloor \frac{x}{n} \right\rfloor + 2\gamma \lfloor x \rfloor$$

and $$\sum_{n \le x} \psi\left(\frac{x}{n}\right) = \sum_{n \le x} \sum_{m \le x/n} \Lambda(m) = \sum_{m \le x} \Lambda(m) \sum_{n \le x/m} 1$$ order of summation * trick

$$= \sum_{m \le x} \Lambda(m) \left\lfloor \frac{x}{m} \right\rfloor = \sum_{\substack{p^k \le x \\ k \ge 1}} (\log p) \left\lfloor \frac{x}{p^k} \right\rfloor$$ (otherwise 0).

$(k=1)+(k=2)+\dots$

$$= \sum_{p \le x} \left(\left\lfloor \frac{x}{p} \right\rfloor + \left\lfloor \frac{x}{p^2} \right\rfloor + \cdots \right) \log p$$

$$= \log(\lfloor x \rfloor !) = \sum_{n \le x} \log n.$$

But (see way back) $$\sum_{n \le x} \log n = x \log x - x + O(\log x) \Rightarrow$$

$$\sum_{n \le x} \psi\left(\frac{x}{n}\right) = x \log x - x + O(\log x)$$ — (3)

Also $$\sum_{n \le x} \left\lfloor \frac{x}{n} \right\rfloor = \sum_{n=1}^{\lfloor x \rfloor} \left\lfloor \frac{\lfloor x \rfloor}{n} \right\rfloor$$ (3')

If $x \in \mathbb{N}$ it is immediate since $x = \lfloor x \rfloor$.

(Lemma

To see (3') is true $\exists m$ so

$mn \le x < (m+1)n \Rightarrow \left\lfloor \frac{x}{n} \right\rfloor = m$

$\exists i$ so $mn \le i \le x < i+1 \le (m+1)n$

$\lfloor x \rfloor = i$ & $mn \le i < (m+1)n \Rightarrow \left\lfloor \frac{i}{n} \right\rfloor = m$.

$$\therefore \sum_{n\le x}\left\lfloor \frac{x}{n}\right\rfloor = \sum_{n=1}^{\lfloor x\rfloor}\left\lfloor \frac{\lfloor x\rfloor}{n}\right\rfloor$$ then by the Proof of Thm 4.9.

$$= \lfloor x\rfloor \log\lfloor x\rfloor + (2\gamma-1)\lfloor x\rfloor + O(\sqrt{\lfloor x\rfloor}).$$

$$= \dots = x\log x - (2\gamma-1)x + O(x^{\frac12}) \quad \text{—— (4)}$$

Therefore (2) + (3) + (4) $\Rightarrow$.

$$F(x) = (x\log x - x + O(\sqrt{x})) - (x\log x + (2\gamma-1)x + O(\sqrt{x})) + 2\gamma x + O(1)$$

$$= O(\sqrt{x}).$$

Hence $\exists\ c>0$ so. $|F(x)| \le c\sqrt{x}$ for. $x \ge x_0$ —— (4')

By choosing a large enough c we can assume $x_0 = 1$.

Now let $t>1$ be an integer. We have

$$\left|\sum_{n<\frac{x}{t}}\mu(n)F\left(\frac{x}{n}\right)\right| \le \sum_{n<\frac{x}{t}}\left|F\left(\frac{x}{n}\right)\right| \le c\sum_{n<\frac{x}{t}}\left(\frac{x}{n}\right)^{\frac12} \quad \left(\text{since } 1<t<\frac{x}{n}\right)$$

$$\le c\sqrt{x}\left(1+\int_1^{\frac{x}{t}}\frac{du}{\sqrt{u}}\right)$$

$$\le c\sqrt{x}\left(1+2\sqrt{u}\Big|_1^{x/t}\right)$$

$$= c\sqrt{x}\left(1+2\sqrt{\frac{x}{t}}-2\right) < \frac{2cx}{\sqrt{t}} \quad \text{—— (5)}$$

Since F is a step function, i.e. constant between integer values of x, $\forall a\in\mathbb{Z}$
$a\le x<a+1 \Rightarrow F(x)=F(a)$. Thus

$$\frac{x}{t} \quad \dots \quad \frac{x}{3} \quad \frac{x}{2} \quad n \quad x$$

$$\sum_{\frac{x}{t}<n\le x}\mu(n)F\left(\frac{x}{n}\right) = F(1)\sum_{\frac{x}{2}<n\le x}\mu(n) + F(2)\sum_{\frac{x}{3}<n\le x/2}\mu(n) + \dots + F(t-1)\sum_{\frac{x}{t}<n\le\frac{x}{t-1}}\mu(n).$$

$$\therefore \left|\sum_{\frac{x}{t}<n\le x}\mu(n)F\left(\frac{x}{n}\right)\right| \le \left(|F(1)|+\dots+|F(t-1)|\right)\max_{2\le j\le t}\left|\sum_{\frac{x}{j}<n\le\frac{x}{j-1}}\mu(n)\right| \quad \text{—— (6)}$$

$$\frac{x}{2}<n\le x \iff 1\le\frac{x}{n}<2$$
$$\frac{x}{3}<n\le\frac{x}{2} \iff 2\le\frac{x}{n}<3.$$

But If $j \geq 2$ is fixed and $x \to \infty$

Thm 5.2 says. $\sum_{n \leq 1}^{\infty} \frac{\mu(n)}{n} = 0 \; (= \alpha)$

Lemma ⊗ implies $\lim_{N \to \infty} \frac{\left|\sum_{n=1}^{N} \mu(n)\right|}{N} = 0$

$\Rightarrow \forall \varepsilon > 0 \; \exists \, x_\varepsilon$ so $\left|\sum_{n=1}^{y} \mu(n)\right| \leq \varepsilon y \quad \forall y \geq x_\varepsilon$.

Then $\left| \sum_{\frac{x}{j} < n \leq \frac{x}{j-1}} \mu(n) \right| \leq \left| \sum_{n \leq \frac{x}{j-1}} \mu(n) - \sum_{n \leq \frac{x}{j}} \mu(n) \right|$

$\leq \left| \sum_{n \leq \frac{x}{j-1}} \mu(n) \right| + \left| \sum_{n \leq \frac{x}{j}} \mu(n) \right| < \varepsilon \left(\frac{x}{j} + \frac{x}{j-1} \right)$

$\leq 2 \varepsilon x$ (7) for $j = t, t-1, \ldots, 2$.

By (6) + (7) + (4')

$\left| \sum_{\frac{x}{t} < n \leq x} \mu(n) F\left(\frac{x}{n}\right) \right| \leq 2 \varepsilon x \sum_{j=1}^{t-1} |F(j)| \leq 2 \varepsilon x \sum_{j=1}^{t-1} c j^{\frac{1}{2}}$

$< 2 c \varepsilon t^{3/2} x$

But then using (5)

$\left| \sum_{n \leq x} \mu(n) F\left(\frac{x}{n}\right) \right| \leq \left| \sum_{n \leq \frac{x}{t}} \mu(n) F\left(\frac{x}{n}\right) \right| + \left| \sum_{x/t < n \leq x} \mu(n) F\left(\frac{x}{n}\right) \right|$

$= |\cdot|$

$< c \left(\frac{2x}{t^{1/2}} + 2 \varepsilon t^{3/2} x \right) = c x \left(\frac{2}{t^{1/2}} + 2 \varepsilon t^{3/2} \right)$

Let $t = \left\lfloor \frac{1}{\sqrt{\varepsilon}} \right\rfloor \geq \frac{1}{2\sqrt{\varepsilon}} \Rightarrow |\cdot| < c x \left(2\sqrt{2}\, \varepsilon^{1/4} + 2 \varepsilon^{1/4} \right)$

$\frac{1}{\varepsilon^{1/2}} \geq$

$< 5 c x \varepsilon^{1/4}$.

$\therefore \frac{\left| \sum_{n \leq x} \mu(n) F\left(\frac{x}{n}\right) \right|}{x} \to 0$ as $x \to \infty$

$\therefore \frac{\psi(x) - \lfloor x \rfloor + 2\gamma}{x} \xrightarrow{(1)} 0 \Rightarrow \frac{\psi(x)}{x} \to 1$ so by Thm 4.2 $\left(\pi(x) \sim \frac{\psi(x)}{\log x} \right)$

$\pi(x) \sim \frac{\psi(x)}{x} \cdot \frac{x}{\log x} \sim \frac{x}{\log x}$ //

p61 #4.1

$\otimes$ $\sum_{n=1}^{\infty} \frac{b_n}{n} = \alpha \Rightarrow \forall \varepsilon > 0 \; \exists N_\varepsilon \in \mathbb{N}$ so. $\forall N \geq N_\varepsilon$

$$\left| \alpha - \sum_{n=1}^{N} \frac{b_n}{n} \right| < \varepsilon \Leftrightarrow \alpha - \varepsilon < \sum_{n=1}^{N} \frac{b_n}{n} < \alpha + \varepsilon.$$

now. Set $a_n = \frac{b_n}{n}$ & $f(t) = t$ in Abel's Summ formula.

so. $A(x) = \sum_{n \leq x} a_n = \sum_{n \leq x} \frac{b_n}{n}$ and $f'(t) \equiv 1$.

$\Rightarrow A(N) = \sum_{n \leq N} \frac{b_n}{n}$. Thus by Abel,

and $\sum_{n \leq N} b_n = \sum_{n \leq N} \frac{b_n}{n} \cdot n = A(N) f(N) - \int_1^N A(t) f'(t) dt$

$$= N A(N) - \int_1^N A(t) dt$$

$A(x)$ is constant over $[n, n+1)$ which has length 1

$$= N A(N) - \sum_{n=1}^{N-1} A(n) \quad \boxtimes$$

Lemma if $c_n \to \beta$ then the average $\frac{c_1 + \dots + c_n}{n} \to \beta$ also.

Proof. Given $\varepsilon > 0 \; \exists N_\varepsilon$ so. $|c_n - \beta| < \varepsilon/2 \quad \forall n \geq N_\varepsilon$.

then $\left| \frac{c_1 + \dots + c_n}{n} - \beta \right| = \left| \frac{(c_1 - \beta) + (c_2 - \beta) + \dots + (c_n - \beta)}{n} \right|$.

$$\leq \frac{\sum_{j=1}^{N_\varepsilon - 1} |c_j - \beta|}{n} + \frac{\sum_{j=N_\varepsilon}^{n} |c_j - \beta|}{n}$$

$$\leq \frac{L}{n} + \frac{\varepsilon}{2} < \varepsilon \text{ provided } n \geq \max\left\{ N_\varepsilon, \frac{2L}{\varepsilon} \right\}.$$

Return to $\boxtimes$ $\frac{\left|\sum_{n \leq N} b_n\right|}{N} = \left| A(N) - \frac{\sum_{n=1}^{N-1} A(n)}{N-1} \left(\frac{N-1}{N}\right) \right| \to 0.$

$A(N) = \sum_{n=1}^{N} \frac{b_n}{n} \to \alpha$, $\frac{\sum_{n=1}^{N-1} A(n)}{N-1} \to \alpha$ (average), $\frac{N-1}{N} \to 1$

and the result follows for x with $N = \lfloor x \rfloor$.

So Lemma $\otimes$ is $\sum_{n=1}^{\infty} \frac{b_n}{n} = \alpha \Rightarrow \lim_{N \to \infty} \frac{\left|\sum_{n \leq N} b_n\right|}{N} = 0.$