Ch 4 Prime Number Theorem Prelude

(PNT) $\pi(x) \sim \frac{x}{\log x}$ as $x \to \infty$.

Conjectured by Gauss, ($\sim$1800)
Proof using $\zeta(s) \neq 0$ on $\sigma \geq 1$ sketched by Riemann (1850)
Proved by Hardamard and de Vallee Poisson using $\zeta(s)$ & $\mathbb{C}$ (1900)
Proof not using $\mathbb{C}$ or $\zeta(s)$ by Erdös and Selberg (1940)

4.1 Von Mangoldt and other functions needed for the proof (Λ, Ψ, θ).

$$\Lambda(n) := \begin{cases} \log p, & n = p^k, \quad k \geq 1, \ p \in \mathbb{P}. \\ 0 & \text{otherwise} \end{cases}$$

$p^k \leq x \iff k \log p \leq \log x$,

so $\Lambda(3^2) = \log 3$, $\Lambda(6) = 0$.

(psi) $$\psi(x) := \sum_{\substack{p^k \leq x \\ k \geq 1}} \log p = \sum_{n \leq x} \Lambda(n) = \sum_{p \leq x} \left\lfloor \frac{\log x}{\log p} \right\rfloor \log p$$

since $p^k \leq x \iff k \leq \frac{\log x}{\log p} \iff k \leq \left\lfloor \frac{\log x}{\log p} \right\rfloor$

(theta) Let $\theta(x) := \sum_{p \leq x} \log p$, $\theta(1) = 0$, $\theta(2) = \log 2, \ldots$

Because $p^k \leq x \iff p \leq x^{1/k}$ ——— (1)

$\psi(x) = \theta(x) + \theta(x^{\frac{1}{2}}) + \ldots + \theta(x^{\frac{1}{m}}) + \ldots$

Need $x^{\frac{1}{m}} \geq 2$ so if $x^{\frac{1}{m}} < 2$ the term $\theta(x^{\frac{1}{m}}) = 0$

$\therefore \frac{1}{m} \log x \geq \log 2 \Rightarrow m \leq \left\lfloor \frac{\log x}{\log 2} \right\rfloor = \#$ terms non 0.

Chebyshev's Thm $\exists \alpha, \beta > 0$ so $\alpha \frac{x}{\log x} \leq \pi(x) \leq \beta \frac{x}{\log x} \Rightarrow$.

$$\theta(x) = \sum_{p \leq x} \log p \leq \sum_{p \leq x} \log x = \pi(x) \log x \leq \beta x \Rightarrow \underline{\theta(x) = O(x)}.$$

Now $\theta(x^{\frac{1}{2}}) \geq \theta(x^{\frac{1}{3}}) \geq \ldots \Rightarrow$

$$\theta(x^{\frac{1}{2}}) + \theta(x^{\frac{1}{3}}) + \ldots \leq \theta(x^{\frac{1}{2}}) \frac{\log x}{\log 2} = O(x^{\frac{1}{2}} \log x)$$

$\therefore$ by (1) $\psi(x) = \theta(x) + O(x^{\frac{1}{2}} \log x)$. ——— (2)

Proposition $\theta(x) \sim \psi(x)$ as $x \to \infty$.

Proof First note that $p \geqslant x^{\frac{1}{2}}$

$$\Rightarrow \log p \geqslant \tfrac{1}{2}\log x \gg \log x$$

$$\therefore \quad \theta(x) \geqslant \sum_{x^{\frac{1}{2}} \leq p \leq x} \log p \gg \left(\pi(x) - \pi(\sqrt{x})\right)\log x$$
$$= \pi(x)\log x - \pi(x^{\frac{1}{2}})\log x$$
$$= \pi(x)\log x + O(x^{\frac{1}{2}}\log x)$$

Because, by Chebyshev, $\frac{x}{\log x} \ll \pi(x)$, we get $\theta(x) \gg x$.

By ② $$\frac{\psi(x)}{\theta(x)} = 1 + O\left(\frac{x^{\frac{1}{2}}\log x}{\theta(x)}\right) = 1 + O\left(\frac{x^{\frac{1}{2}}\log x}{x}\right) = 1 + O\Big(\underbrace{\tfrac{\log x}{\sqrt{x}}}_{\to 0}\Big)$$

$$\therefore \quad \psi(x) \sim \theta(x) \text{ as } x \to \infty.$$

Theorem 4.2 $$\pi(x) \sim \frac{\theta(x)}{\log x} \sim \frac{\psi(x)}{\log x} \text{ as } x \to \infty.$$

Proof The final $\sim$ follows from the proposition above.

So first $$\theta(x) = \sum_{p \leq x} \log p \leq \log x \sum_{p \leq x} 1 = \pi(x)\log x$$

$$\Rightarrow \quad \frac{\theta(x)/\log x}{\pi(x)} \leq 1 \quad ④$$

Let $0 < \delta < \frac{1}{2}$ be a small fixed constant so $x^{1-\delta} < x$ and

$$\theta(x) \geqslant \sum_{x^{1-\delta} < p \leq x} \log p > \log(x^{1-\delta}) \sum_{x^{1-\delta} < p \leq x} 1 = (1-\delta)\log x\left(\pi(x) - \pi(x^{1-\delta})\right)$$
$$= (1-\delta)\pi(x)\log x + O(x^{1-\delta}) \quad - ③$$

again using Chebyshev to get the $O(\cdot)$ with constant $\beta > 0$,

Let $\pi(x)\log x > \beta x$. By ③

$$1 \overset{④}{\geqslant} \frac{\theta(x)/\log x}{\pi(x)} \geqslant (1-\delta) - \frac{\alpha x^{1-\delta}}{\pi(x)\log x} \geqslant (1-\delta) - \frac{\alpha}{\beta x^{\delta}}$$

$\therefore$ letting $x \to \infty$, $$\frac{\theta(x)/\log x}{\pi(x)} \to 1 \Rightarrow \pi(x) \sim \frac{\theta(x)}{\log x} \quad //$$

§4.3 Exploring $\Lambda(n)$, Von Mangoldt

Theorem 4.3 $$\sum_{n \le x} \frac{\Lambda(n)}{n} = \log x + O(1)$$

Proof In Abel Summation (Prop 1.4) let $a_n = 1$ and $f(t) = \log t$ so

$$\sum_{\substack{1 \le n \le x \\ 1 < n \le x}} 1 . \log n = \lfloor x \rfloor \log x - \lfloor 1 \rfloor \log 1 - \int_1^x \frac{\lfloor t \rfloor}{t} dt$$

$$= (x - \{x\}) \log x - \int_1^x \frac{t - \{t\}}{t} dt$$

$$= x \log x + O(\log x) - \int_1^x 1 dt + \int_1^x \frac{\{t\}}{t} dt$$

$$= x \log x + O(\log x) - (x-1) + O\left(\int_1^x \frac{dt}{t}\right)$$

$$= x \log x - x + O(\log x). \quad \text{——— (5)}$$

But $$\nu_p(N!) = \sum_{j=1}^{\infty} \left\lfloor \frac{N}{p^j} \right\rfloor .$$ Therefore

$$\sum_{1 \le n \le x} \log n = \log(\lfloor x \rfloor !) = \log\left(\prod_{p \le N} p^{\nu_p(N!)}\right)$$

$N = \lfloor x \rfloor$

$$= \sum_{p \le x} \left(\sum_{j=1}^{\infty} \left\lfloor \frac{x}{p^j} \right\rfloor \right) \log p$$

$$= \sum_{\substack{p^k \le x \\ k \ge 1}} \left\lfloor \frac{x}{p^k} \right\rfloor \log p = \sum_{n \le x} \left\lfloor \frac{x}{n} \right\rfloor \Lambda(n).$$

$$= \sum_{n \le x} \frac{x}{n} \Lambda(n) - \sum_{n \le x} \left\{ \frac{x}{n} \right\} \Lambda(n).$$

$$= x \sum_{n \le x} \frac{\Lambda(n)}{n} - O\Big(\underbrace{\sum_{n \le x} \Lambda(n)}_{= \psi(x) = O(x)}\Big)$$

See back
$O(x)$
$\theta(x) \sim \psi(x)$

$$\Rightarrow \sum_{1 \le n \le x} \log n = x \sum_{n \le x} \frac{\Lambda(n)}{n} + O(x) \quad \text{——— (6)}$$

Equating (5) + (6) and dividing by x gives:

$$\log x - 1 + O\left(\frac{\log x}{x}\right) = \sum_{n \le x} \frac{\Lambda(n)}{n} + O\left(\frac{x}{x}\right)$$

$$\Rightarrow \sum_{n \le x} \frac{\Lambda(n)}{n} = \log x + O(1) \quad //$$

Theorem 4.4 $\displaystyle\sum_{p\le x} \frac{\log p}{p} = \log x + O(1)$ as $x \to \infty$.

Proof
$$\sum_{p\le x} \frac{\log p}{p} = \sum_{n\le x}\frac{\Lambda(n)}{n} - \underbrace{\sum_{k\ge 2}\sum_{p^k\le x} \frac{\log p}{p^k}}_{S}$$
$$\overset{\text{Tm 4.3}}{=} \log(x) + O(1) - S$$

But
$$S \le \sum_p \left(\frac{1}{p^2}+\frac{1}{p^3}+\dots\right)\log p = \sum_p \frac{\log p}{p(p-1)}$$
$$\le \sum_{n\ge 1}\frac{2\log n}{n^2} = O(1)$$

So the theorem follows. //

Theorem 4.5 [Merten's] As $x\to\infty$
$$\sum_{p\le x}\frac{1}{p} = \log\log x + \beta + O\left(\frac{1}{\log x}\right) \qquad (\beta = 0.26149\ldots)$$

where $\beta > 0$ is "merten's constant".

Note: Here $\displaystyle\sum_p \frac{1}{p} = \infty$. note $\displaystyle\sum_{n\ge 1}\frac{1}{n^2} = \frac{\pi^2}{6} < \infty$ so the primes are "more plentiful" than squares.

Proof Again by Abel Summation. $a_n := \begin{cases} \frac{\log p}{p} & \text{if } p\in\mathbb{P} \\ 0 & n\notin\mathbb{P}.\end{cases}$

$f(t) := \frac{1}{\log t} \Rightarrow f'(t) = -\frac{1}{t\log^2 t}$.

Uses $\displaystyle A(x) = \sum_{n\le x} a_n = \sum_{p\le x}\frac{\log p}{p} = \log x + O(1)$
$= \log x + R(x)$ i.e. $R(x) := A(x) - \log x$.

Then
$$\sum_{2\le p\le x}\frac{1}{p} = \sum_{n\le x} a_n f(n) = \frac{A(x)}{\log x} + \int_2^x \frac{A(t)}{t\log^2 t}\,dt$$
$$= 1 + \frac{R(x)}{\log x} + \int_2^x \frac{\log t + R(t)}{t\log^2 t}\,dt \qquad \int_2^\infty \frac{R(t)}{t\log^2 t}\,dt + O\left(\int \frac{dt}{t\log^2 t}\right) = \ldots$$
$$= 1 + O\left(\frac{1}{\log x}\right) + \int_2^x \frac{dt}{t\log t} + \int_2^\infty \frac{R(t)}{t\log^2 t}\,dt + O\left(\int \frac{dt}{t\log^2 t}\right)$$
$$= \log\log x + \beta + O\left(\frac{1}{\log x}\right) //$$

4.5, 4.6 Arithmetic Functions (μ, $d = \tau$.)

Möbius function $\mu : \mathbb{N} \to \{-1, 0, 1\}$.

$\mu(1) = 1$

$\mu(n) = (-1)^r$ if $n = p_1^{\alpha_1} \dots p_r^{\alpha_r}$, $\alpha_i = 1$ distinct factors

i.e. n squarefree

$\mu(n) = 0$ otherwise.

$\mu(2.3) = (-1)^2 = 1$

$\mu(p) = -1 \quad \forall p \in \mathbb{P}$, $\mu(12) = 0$.

A useful but weird function.

$(a,b) = 1 \implies \mu(ab) = \mu(a) \cdot \mu(b)$ multiplicative.

note $\Lambda(n)$ is not multiplicative

Thm 4.8 (i) $n > 1 \implies \sum_{d|n} \mu(d) = 0$.

Proof $n = p_1^{\alpha_1} \dots p_r^{\alpha_r}$ $\alpha_i \geq 1$, distinct primes p_i $1 \leq i \leq r$

Let $m = p_1 \dots p_r$ the squarefree core of n.

Then $\sum_{d|n} \mu(d) = \sum_{d|m} \mu(d) = 1 - \binom{r}{1} + \binom{r}{2} - \dots + (-1)^r \binom{r}{r} = (1-1)^r = 0$ //

4.8 (iii) Let $f : \mathbb{N} \to \mathbb{C}$ be any function

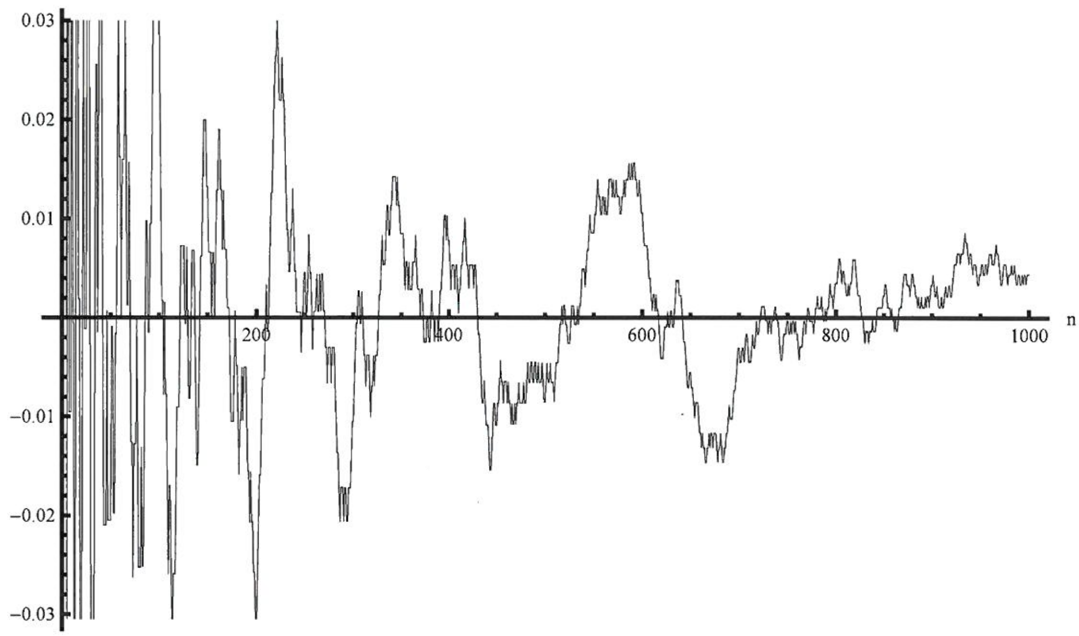
Let $F(n) = \sum_{d|n} f(d)$ (a so-called "divisor sum".)

then $f(n) = \sum_{d|n} \mu(d) F\left(\frac{n}{d}\right)$ (möbius inversion)

Proof
$$f(n) = \sum_{c|n} \Big(\sum_{d|\frac{n}{c}} \mu(d) \Big) f(c)$$
$$= \sum_{d|n} \sum_{c|\frac{n}{d}} \mu(d) f(c) = \sum_{d|n} \mu(d) \sum_{c|\frac{n}{d}} f(c)$$
$$= \sum_{d|n} \mu(d) F\left(\frac{n}{d}\right). //$$

In[3102]:= ListLinePlot[Table[Sum[MoebiusMu[n] / n, {n, 1, m}], {m, 1, 1000}],
ImageSize → 500, AxesLabel → {"n", ""}, AxesStyle → Thick]

Out[3102]=



Let $d(n) = \sum_{d|n} 1 =$ # of divisors of n

e.g. $n = 12$, $\mathcal{D} = \{1, 2, 3, 4, 6, 12\}$ so $d(12) = 6$

Ex $n = p_1^{\alpha_1} \cdots p_r^{\alpha_r} \Rightarrow d(n) = \prod_{j=1}^{r} (1 + \alpha_j)$

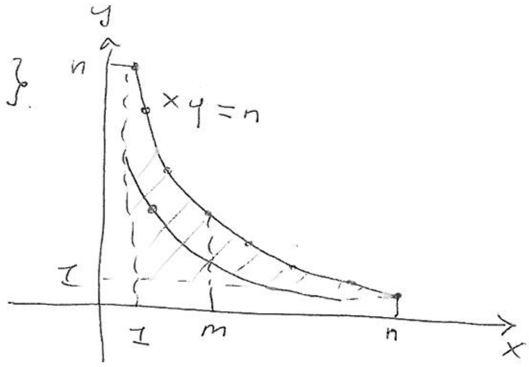
$\Rightarrow$ d is multiplicative, $(a,b) = 1 \Rightarrow d(ab) = d(a) \cdot d(b)$.

Thm 4.9 $$\sum_{m=1}^{n} d(m) = \sum_{m=1}^{n} \left\lfloor \frac{n}{m} \right\rfloor = n \log n + (2\gamma - 1)n + O(\sqrt{n}).$$

Euler's Const.

Proof

$D_n = \{(x,y) \in \mathbb{N}^2 : x.y \le n\}$.

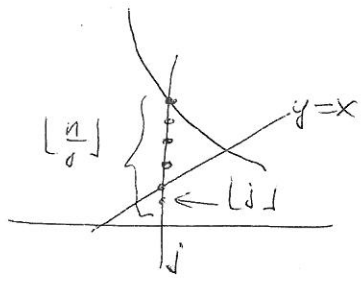


then $\sum_{m=1}^{n}$ corresponds to the number of points with integer coordinates in the hyperbolic region in the first quadrant and this is $\sum_{m=1}^{n} \left\lfloor \frac{n}{m} \right\rfloor$

$= \#(\text{points } \underline{\text{above}}\ y = x) + \#(\text{points } \underline{\text{on}}\ y = x) + \#(\text{points below } y = x)$.

$$= 2 \sum_{j=1}^{\lfloor \sqrt{n} \rfloor} \left(\left\lfloor \frac{n}{j} \right\rfloor - \lfloor j \rfloor \right) + \lfloor \sqrt{n} \rfloor$$

$$= \circ\circ\circ = n \log n + (2\gamma - 1)n + O(\sqrt{n})$$



using $$\sum_{1 \le k \le x} \frac{1}{k} = \log x + \gamma + O\left(\frac{1}{x}\right)$$ //

Read: History & recent result p61.

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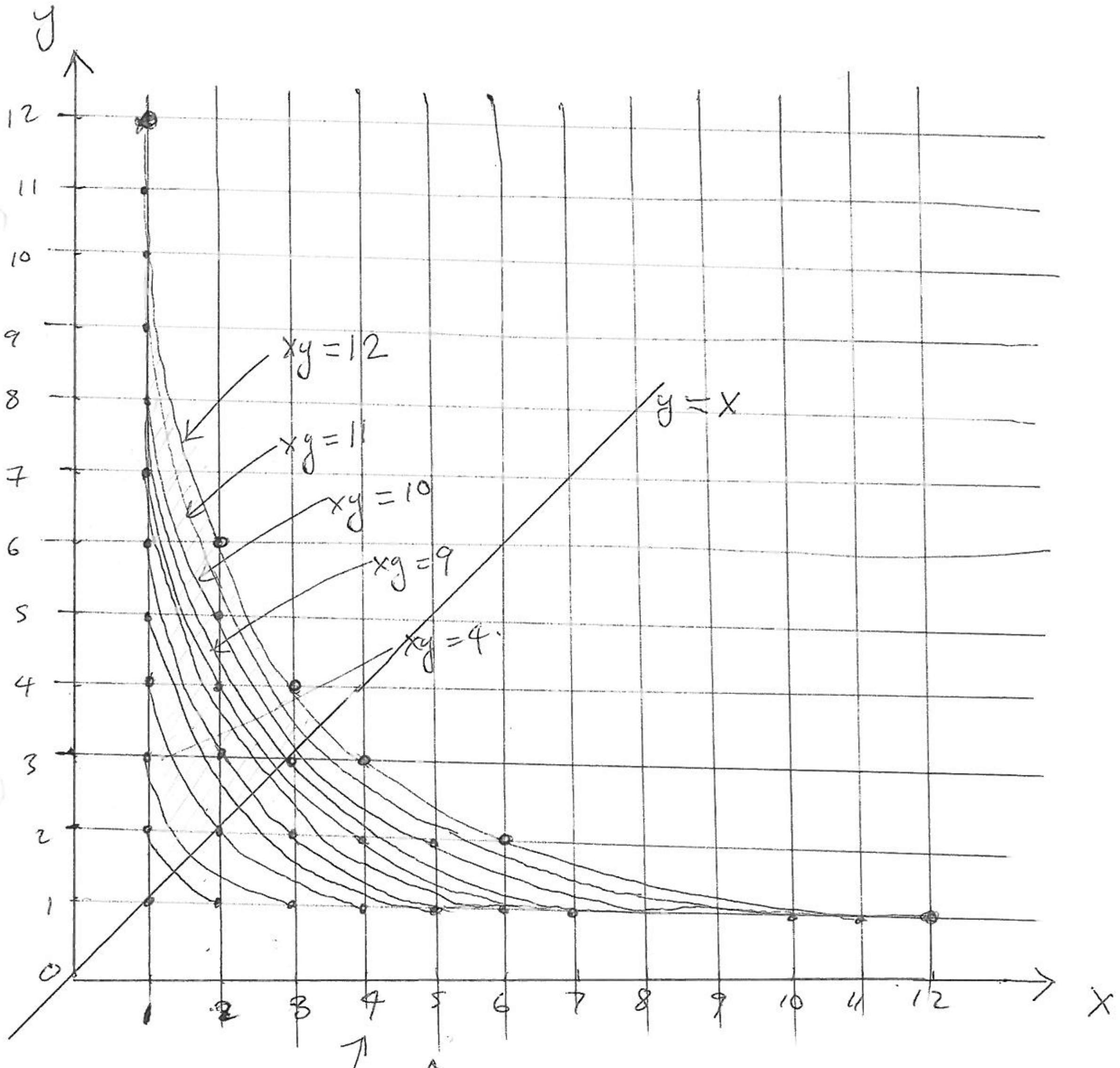
$$\sum_{1 \le n \le 12} d(n) = \sum_{1 \le m \le 12} \left\lfloor \frac{12}{m} \right\rfloor$$



$3 = \left\lfloor \frac{12}{4} \right\rfloor$ $\left\lfloor \frac{12}{5} \right\rfloor = 2$