

Let

$$\text{Li}(x) = \int_2^x \frac{dt}{\log(t)}$$

for  $x \geq 2$  be the “logarithmic integral” and  $\pi(x) = \#\{p \in \mathbb{P} : 2 \leq p \leq x\}$ . Consider the following data:

So  $\pi(x) \doteq \frac{x}{\log(x)}$  but  $\pi(x) \doteq \text{Li}(x)$  is better, and  $\frac{\pi(x)}{x} \rightarrow 0$  as  $x \rightarrow 0$  apparently. Indeed  $\pi(x) \sim \frac{x}{\log(x)} \sim \text{Li}(x)$ .

This distribution is the subject of the famous **Prime Number Theorem**, which took all of the 19<sup>th</sup> century to prove.

Because  $\log(10^n) = n \log(10)$

in  $[2, 100]$  :                      about  $\frac{1}{2}$  the numbers are prime  
 in  $[2, 1000]$  :                     $\frac{1}{3}$   
 in  $[2, 1,000,000]$  :             $\frac{1}{6}$  etc.

so they progressively thin out with a *local density*  $\frac{1}{\log(t)}$  since if  $a < b$

$$\#\{p \in \mathbb{P} : a \leq p \leq b\} = \pi(b) - \pi(a) \sim \int_2^b \frac{dt}{\log t} - \int_2^a \frac{dt}{\log t} = \int_a^b \frac{dt}{\log(t)}.$$

**Theorem 20** For  $n \geq 2$ ,

$$\frac{1}{8} \leq \frac{\pi(n)}{n/\log n} \leq 12.$$

**Note:** This is as close as we will get to proving the Prime Number Theorem.

**Lemma (Chebyshev)** If  $H(n) = \sum_{j=2}^n \frac{1}{j}$  then

$$\frac{1}{8} \leq \pi(n) \frac{H(n)}{n} \leq 6.$$

*Proof.* Proof of Theorem 20 assuming Chebyshev's Lemma:

For  $n \geq 2$ ,

$$\log\left(\frac{n}{2}\right) = \int_2^n \frac{dt}{t} < \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n} < \int_1^n \frac{dt}{t} = \log(n).$$

For  $n \geq 4$ ,

$$\frac{1}{2} \log(n) \leq \log\left(\frac{n}{2}\right).$$

Hence  $\frac{1}{2} \log(n) \leq H(n) \leq \log(n)$  so, by the RHS of Chebyshev's Lemma,

$$\frac{\pi(n) \log(n)}{n} \leq \frac{\pi(n) 2H(n)}{n} \leq 12$$

and by the LHS of Chebyshev's Lemma

$$\frac{1}{8} \leq \frac{\pi(n) H(n)}{n} \leq \frac{\pi(n) \log(n)}{n}$$

using Lemma 2 when  $n \geq 4$ .

If  $n = 2$ ,  $\pi(2) = 1$  and

$$\frac{1}{8} \leq \underbrace{\frac{1}{2/\log(2)}}_{0.34} \leq 6.$$

If  $n = 3$ ,  $\pi(3) = 2$  and

$$\frac{1}{8} \leq \underbrace{\frac{2}{3/\log(3)}}_{0.73} \leq 6.$$

This completes the proof of the theorem.

*Proof of Lemma 2:*

$$\text{Claim : } \forall k \geq 0, \pi(2^{k+1}) \leq 2^k \quad (9)$$

*Proof:* If  $x > 9$ ,  $\pi(x) \leq \frac{x}{2}$  since all even numbers greater than 2 are composite. Since  $\pi(2^1) = 1 = 2^0$ ,  $\pi(4) = 2 = 2^1$  and  $\pi(8) = 4 = 2^2$ , (1) is true  $\forall k \geq 0$ .

$$\text{Claim : } \frac{1}{2} \ell \leq H(2^\ell) \leq \ell \quad (10)$$

where  $H(n) = \frac{1}{2} + \dots + \frac{1}{n}$ .

$$\begin{aligned}
 H(2^\ell) &= \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \dots + \left(\frac{1}{2^{\ell-1} + 1} + \dots + \frac{1}{2^\ell}\right) \\
 &\geq \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) + \dots + \left(\frac{1}{2^\ell} + \dots + \frac{1}{2^\ell}\right) \\
 &= \frac{\ell}{2}
 \end{aligned}$$

and

$$\begin{aligned}
 H(2^\ell) &= \left(\frac{1}{2} + \frac{1}{3}\right) + \left(\frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7}\right) + \left(\dots + \frac{1}{2^\ell}\right) \\
 &\leq \left(\frac{1}{2} + \frac{1}{2}\right) + \left(\frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4}\right) + \dots + \left(\frac{1}{2^{\ell-1}} + \dots + \frac{1}{2^{\ell-1}}\right) \\
 &\leq \ell
 \end{aligned}$$

This proves the claim.

If  $p \in \mathbb{P}$  has  $n < p < 2n \Rightarrow p | 2n!$  and  $p \nmid n! \Rightarrow$

$$p \mid \binom{2n}{n} = \frac{2n!}{n!n!} \Rightarrow \prod_{n < p < 2n} p \mid \binom{2n}{n} \quad (11)$$

By Lagrange, the power of  $p$  in  $\binom{2n}{n}$  is

$$\sum_{m=1}^r \left( \left\lfloor \frac{2n}{p^m} \right\rfloor - 2 \left\lfloor \frac{n}{p^m} \right\rfloor \right) \quad (12)$$

where  $p^r \leq 2n < p^{r+1}$  and the sum is  $\leq r$  since  $\forall x, \lfloor 2x \rfloor - 2 \lfloor x \rfloor \leq 1$  (See below). Hence

$$\binom{2n}{n} \leq \prod_{p^r \leq 2n < p^{r+1}} p^r$$

By (11) and (12)

$$n^{\pi(2n) - \pi(n)} < \prod_{n < p < 2n} p \leq \binom{2n}{n} \leq \prod_{p^r \leq 2n < p^{r+1}} p^r \leq (2n)^{\pi(2n)} \quad (13)$$

Now

$$\binom{2n}{n} \leq (1+1)^{2n} = 2^{2n}$$

and

$$\binom{2n}{n} = \frac{2n(2n-1)\cdots(n+1)}{n(n-1)\cdots 1} = 2 \left( 2 + \frac{1}{n-1} \right) \left( 2 + \frac{2}{n-2} \right) \cdots \left( 2 + \frac{n-1}{1} \right) \geq 2^n$$

so

$$2^n \leq \binom{2n}{n} \leq 4^n \quad (14)$$

Using LHS of (13) we get  $n^{\pi(2n)-\pi(n)} < 2^{2n}$  and the RHS gives  $2^n < (2n)^{\pi(2n)}$ ,  $n \geq 1$ .  
Now let  $n = 2^k$ ,  $k = 0, 1, 2, \dots$  so these two inequalities translate to

$$2^{k(\pi(2^{k+1})-\pi(2^k))} \leq 2^{2^{k+1}}, \quad 2^{2^k} \leq 2^{(k+1)\pi(2^{k+1})}, \quad k \geq 0$$

or

$$k(\pi(2^{k+1}) - \pi(2^k)) \leq 2^{k+1}, \quad 2^k \leq (k+1)\pi(2^{k+1}). \quad (15)$$

Hence

$$\begin{aligned} (k+1)\pi(2^{k+1}) - k\pi(2^k) &= k(\pi(2^{k+1}) - \pi(2^k)) + \pi(2^{k+1}) \\ &\leq 2^{k+1} + \pi(2^{k+1}) \\ &< 2^{k+1} + 2^k \quad \text{by (9)} \\ &= 3 \cdot 2^k \end{aligned}$$

Apply this for  $k = 0, 1, 2, \dots, k$  and add ( $\pi(2^0) = \pi(1) = 0$ ):

$$\Rightarrow (k+1)\pi(2^{k+1}) < 3(2^0 + 2^1 + \dots + 2^k) < 3 \cdot 2^{k+1}. \quad (16)$$

By (15) and (16),  $\forall k \geq 0$

$$\frac{1}{2k+1} \frac{2^{k+1}}{2k+1} \leq \pi(2^{k+1}) < 3 \frac{2^{k+1}}{k+1}.$$

If  $n \in \mathbb{N}$ ,  $n > 1$  choose  $k$  so  $2^{k+1} \leq n < 2^{k+2}$ . By (10) ( $\pi$  is increasing)

$$\pi(n) \leq \pi(2^{k+2}) < 3 \frac{2^{k+2}}{k+2} \leq 6 \frac{2^{k+1}}{H(2^{k+2})} \leq \frac{6n}{H(n)}$$

( $H$  is increasing) and

$$\begin{aligned} \pi(n) &\geq \pi(2^{k+1}) \geq \frac{1}{2} \frac{2^{k+1}}{k+1} \\ &= \frac{1}{8} \frac{2^{k+2}}{k+1} \\ &\geq \frac{1}{8} \frac{2^{k+2}}{H(2^{k+1})} \\ &\geq \frac{1}{8} \frac{1}{n} \\ &\geq \frac{1}{8} \frac{1}{H(n)} \end{aligned}$$

$$\Rightarrow \frac{1}{8} \leq \frac{\pi(n)}{n/H(n)} \leq 6$$

as claimed.  $\square$

**Ex**  $\forall x \in \mathbb{R}$ ,  $0 \leq \lfloor 2x \rfloor - 2 \lfloor x \rfloor \leq 1$

*Proof.*

$$\begin{aligned} \lfloor x \rfloor &\leq x \Rightarrow 2 \lfloor x \rfloor \leq 2x \\ \text{and } 2 \lfloor x \rfloor &\in \mathbb{Z} \therefore 2 \lfloor x \rfloor \leq \lfloor 2x \rfloor \end{aligned}$$

$$\text{so } 0 \leq \lfloor 2x \rfloor - 2 \lfloor x \rfloor$$

If  $x \in \mathbb{Z}$ , then  $\lfloor 2x \rfloor - 2 \lfloor x \rfloor = 2x - 2 \cdot x = 0 \leq 1$ .

If  $x \notin \mathbb{Z}$ ,  $\exists n \in \mathbb{Z}$  so  $n < x < n+1$  and  $x = n + \frac{1}{2} + \varepsilon$  where  $|\varepsilon| < \frac{1}{2}$ . Then  $2x = 2n + 1 + 2\varepsilon$ .

$$\begin{aligned} \lfloor 2x \rfloor &= \lfloor 2n + 1 + 2\varepsilon \rfloor \\ &= 2n + 1 + \lfloor 2\varepsilon \rfloor & \lfloor x \rfloor &= n \\ &= 2n + 1 \end{aligned}$$

Hence  $\lfloor 2x \rfloor - 2 \lfloor x \rfloor = (2n + 1) - 2n = 1 \leq 1$ .  $\square$

Note we have used several times the result

$$\lfloor y + n \rfloor = \lfloor y \rfloor + n \quad \forall n \in \mathbb{Z}. \quad (\text{Ex})$$

## 5 Primes in Gaps

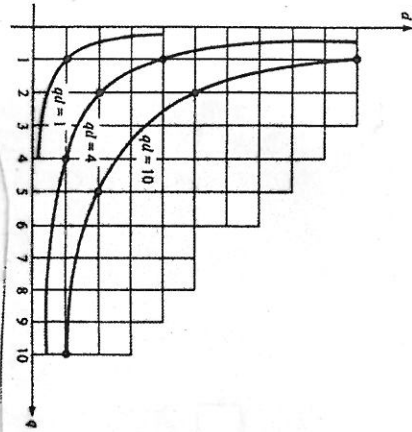
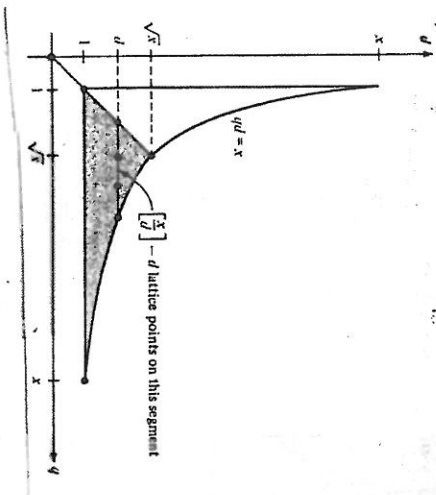
- primes can be *close* together:  $\{11, 13\}$ ,  $\{29, 31\}$ ,  $\{101, 103\}, \dots$

- there can be *long* stretches of  $\mathbb{N}$  with *no primes*:

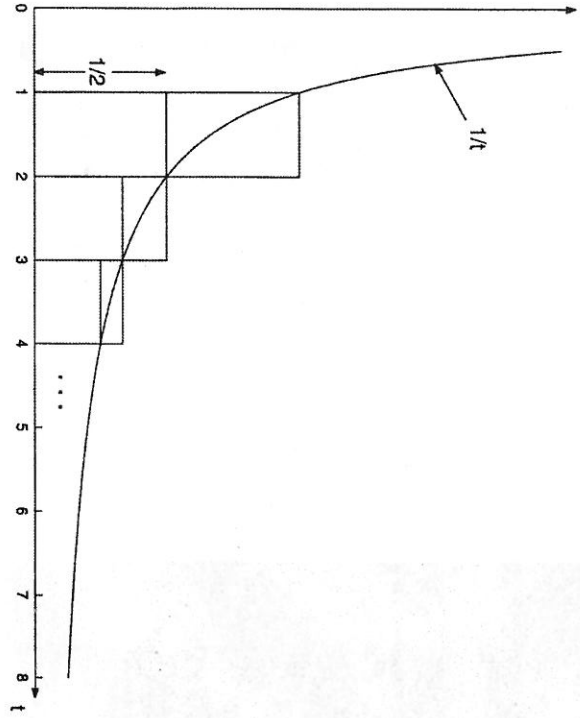
$$\left. \begin{array}{l} a_1 = n! + 2 \\ a_2 = n! + 3 \\ \vdots \\ a_{n-1} = n! + n \end{array} \right\} \begin{array}{l} \text{are } n - 1 \text{ composite and consecutive} \\ \text{numbers, so none are prime} \\ \text{and } n \text{ can be as large as you like.} \end{array}$$

- we will prove the celebrated **Bertrand's Hypothesis**:  $\forall n \in \mathbb{N}, \exists p \in \mathbb{P}$  with  $n \leq p < 2n$ .

- $\forall n \in \mathbb{N}$  does there exist a  $p \in \mathbb{P}$  with  $n^2 < p < (n + 1)^2$ ?



$x$	$n(x)$	$\log x$	$h_x$	$F_x$	$n(x)$	$x$
1.00	168	1.00	1.76	0.94	0.1680	1.00
1.10	142	1.04	1.76	0.93	0.1729	1.10
1.20	118	1.08	1.76	0.93	0.1826	1.20
1.30	99	1.12	1.76	0.93	0.1900	1.30
1.40	84	1.15	1.76	0.93	0.1959	1.40
1.50	73	1.18	1.76	0.93	0.2008	1.50
1.60	64	1.21	1.76	0.93	0.2051	1.60
1.70	57	1.23	1.76	0.93	0.2089	1.70
1.80	51	1.25	1.76	0.93	0.2123	1.80
1.90	46	1.27	1.76	0.93	0.2153	1.90
2.00	42	1.30	1.76	0.93	0.2179	2.00
2.50	31	1.40	1.76	0.93	0.2303	2.50
3.00	23	1.48	1.76	0.93	0.2400	3.00
4.00	15	1.60	1.76	0.93	0.2570	4.00
5.00	10	1.70	1.76	0.93	0.2709	5.00
6.00	7	1.78	1.76	0.93	0.2818	6.00
7.00	5	1.85	1.76	0.93	0.2900	7.00
8.00	4	1.90	1.76	0.93	0.2959	8.00
10.00	3	2.00	1.76	0.93	0.3085	10.00
15.00	2	2.18	1.76	0.93	0.3271	15.00
20.00	1	2.30	1.76	0.93	0.3401	20.00
30.00	1	2.48	1.76	0.93	0.3570	30.00
40.00	1	2.60	1.76	0.93	0.3700	40.00
50.00	1	2.70	1.76	0.93	0.3800	50.00
60.00	1	2.78	1.76	0.93	0.3880	60.00
70.00	1	2.85	1.76	0.93	0.3940	70.00
80.00	1	2.90	1.76	0.93	0.3980	80.00
90.00	1	2.95	1.76	0.93	0.4010	90.00
100.00	1	3.00	1.76	0.93	0.4030	100.00
1000.00	1	3.00	1.76	0.93	0.4030	1000.00



Primes in Gaps  
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John Butcher taught a course in number theory soon after being appointed to the University of Auckland and I was a student in that class. An abiding interest in prime numbers has been a result of this course. In this talk I will give the elementary proof of Erdős that there is a prime in every gap:

$$n \leq p \leq 2n$$

describe some large gaps and report on progress that has been made towards proving the difficult conjecture that there is a prime in every gap:

$$n^2 \leq p \leq (n+1)^2$$

as well, of course, as giving some numerical evidence for its potential correctness.

Clps are consecutive composites: (40)

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Searching for early jobs:

41

$g$	$p(g)$
456	25056082543
464	42652618807
468	127976385139
474	182226896713
486	241160624629
490	297501076289
500	303371455741
514	304599509051
516	416608696337
532	461690510543
534	614487454057
540	738832928467
582	1346294311331
588	1408695494197
602	1968188557063
652	2614941711251
674	7177162612387
716	13829048560417
766	19581334193189
778	42842283926129
804	90874329412297

←

1993

Aron, Potter, Young

$$\lim_{n \rightarrow \infty} [\text{size of gap } n] = \infty.$$

If  $n \in \mathbb{N}$

$$a_1 = (n+1)! + 2$$

$$a_2 = (n+1)! + 3$$

$$a_3 = (n+1)! + 4$$

⋮

$$a_n = (n+1)! + (n+1)$$

Then  $\{a_1, \dots, a_n\}$  are consecutive and  $i+1 \mid a_i \Rightarrow$  composite.

But [1993] best gap length = 804 at  $p \approx 10^{15}$ .

$$n = 804 \Rightarrow n! \approx 0.771 \times 10^{1977}.$$

### Proof of Bertrand's Postulate

*Proof. Claim:*

$$x \geq 2 \Rightarrow \prod_{p \leq x} p \leq 4^{x-1} \quad (17)$$

If  $q$  is the largest prime less than or equal to  $x$

$$\prod_{p \leq x} p = \prod_{p \leq q} p \text{ and } 4^{q-1} \leq 4^{x-1}$$

so we can assume  $x = q$  is *prime*. If  $q = 2$ ,  $2 \leq 4^{2-1}$  so let  $q = 2m + 1$  be *odd*. Then

$$\begin{aligned} \prod_{p \leq 2m+1} p &= \left( \prod_{p \leq m+1} p \right) \cdot \left( \prod_{m+1 < p \leq 2m+1} p \right) \\ &= A \cdot B \end{aligned}$$

By induction  $A \leq 4^m$ . Also

$$\binom{2m+1}{m} = \frac{(2m+1)!}{m!(m+1)!}$$

so all primes in  $B$  divide the numerator and are *not cancelled* so

$$B \leq \binom{2m+1}{m} = \binom{2m+1}{m+1} \leq \frac{1}{2}(1+1)^{2m+1}$$

Hence  $A \cdot B \leq 4^m 2^{2m} = 4^{2m+1-1} = 4^{x-1}$ , which proves the claim.

### Legendre's Theorem Implications

$n!$  contains the prime factor  $p$  exactly  $\sum_{j \geq 1} \left\lfloor \frac{n}{p^j} \right\rfloor$  times.

Ex  $24! = 2^{22} \cdot 3^{10} \cdot 5^4 \cdot 7^3 \cdot 11^2 \cdot 13 \cdot 17 \cdot 19 \cdot 23$

$$p = 23 : \left\lfloor \frac{24}{23} \right\rfloor = 1, \left\lfloor \frac{24}{23^2} \right\rfloor = 0, \dots$$

$$p = 7 : \left\lfloor \frac{24}{7} \right\rfloor = 3, \left\lfloor \frac{24}{7^2} \right\rfloor = 0, \dots$$

*Claim:*

$$p^r \mid \binom{2n}{n} \Rightarrow p^r \leq 2n.$$

$\binom{2n}{n}$  contains  $p \sum_{j \geq 1} \left( \left\lfloor \frac{2n}{p^j} \right\rfloor - 2 \left\lfloor \frac{n}{p^j} \right\rfloor \right)$  times. But

$$0 \leq \left\lfloor \frac{2n}{p^j} \right\rfloor - 2 \left\lfloor \frac{n}{p^j} \right\rfloor < \frac{2n}{p^j} - 2 \left( \frac{n}{p^j} - 1 \right) = 2$$

$\Rightarrow$  each summand is 0 or 1 and is 0 for  $p^j > 2n$

$$\Rightarrow \sum_{j \geq 1} \left( \left\lfloor \frac{2n}{p^j} \right\rfloor - 2 \left\lfloor \frac{n}{p^j} \right\rfloor \right) \leq \max\{j : p^j \leq 2n\}$$

$\Rightarrow$  if  $p^2 > 2n$ ,  $p$  occurs at most once in  $\binom{2n}{n}$ .

If  $\frac{2}{3}n < p \leq n \Rightarrow p$  does not appear in  $\binom{2n}{n}$ .

$\frac{2}{3}n < p \Rightarrow 2n < 3p \Rightarrow p, 2p$  are the only multiples of  $p$  in the numerator of  $\frac{(2n)!}{n!n!}$ .  
 $p \leq n \Rightarrow$  there are *two* in the denominator. So they *cancel*.

**Ex**  $n = 24$

$$\binom{48}{24} = 2^2 \cdot 3^2 \cdot 5^2 \cdot 13 \cdot 29 \cdot 31 \cdot 37 \cdot 41 \cdot 43 \cdot 47$$

$$\sqrt{2 \times 24} \quad 16 < p \leq 24$$

### Grand Finale

Assume that for some  $n \in \mathbb{N}$  there is *no*  $p$  in  $n < p < 2n$  (???). Now

$$\sum_{j=0}^{2n} \binom{2n}{j} = 2^{2n} \Rightarrow \binom{2n}{n} \geq \frac{4^n}{2n}$$

Hence

$$\frac{4^n}{2n} \leq \binom{2n}{n} \leq \underbrace{\left( \prod_{p \leq \sqrt{2n}} 2n \right)}_A \cdot \underbrace{\left( \prod_{\sqrt{2n} < p \leq \frac{2}{3}n} p \right)}_B \cdot \underbrace{\left( \prod_{n < p \leq 2n} p \right)}_C$$

$A \leq (2n)^{\sqrt{2n}}$ ,  $C = 1$  by (??),  $B \leq 4^{\frac{2}{3}n}$  by (17). Thus

$$4^n \leq (2n)^{1+\sqrt{2n}} 4^{\frac{2n}{3}}$$

$$\Rightarrow \frac{2}{3} \log(4) \leq (1 + \sqrt{2n}) \log(2n) \Rightarrow n < 468.$$

But  $n < p < 2n \Leftrightarrow p_{n+1} < 2p_n$ .

Consider the primes  $q_j$

{2, 3, 5, 7, 13, 23, 43, 83, 163, 317, 631}

$q_{j+1} < 2q_j$  so Bertrand's Postulate is true for  $n < 468$  (!!!), hence it is true  $\forall n \geq 2$ .  $\square$

### Prime Number Theorem Implications

$$\lim_{x \rightarrow \infty} \frac{\pi(x) \log(x)}{x} = 1$$

or

$$\pi(x) = \frac{x}{\log(x)} + o\left(\frac{x}{\log(x)}\right)$$

Number of primes in  $(x, x(1 + \varepsilon)]$ ,  $\varepsilon > 0$ , is

$$\pi(x + \varepsilon x) - \pi(x) = \frac{\varepsilon x}{\log(x)} + o\left(\frac{x}{\log(x)}\right) > 0 \quad \text{for } x \geq x_\varepsilon$$

$\Rightarrow \exists p$  with  $x < p \leq (1 + \varepsilon)x$  Let  $\varepsilon = 1 \Rightarrow \forall n \geq N_2 \exists p$

$$n < p < 2n$$

Bertrand's Postulate

Chebyshev [1850]

Ramanujan

Erdős at age 19 years

### Progress beyond Bertrand

$$\exists \theta < 1 \text{ with } \pi(x + x^\theta) - \pi(x) \sim \frac{x^\theta}{\log(x)}$$

1930	Hohelsel	$\theta = 1 - \frac{1}{33,000} + \varepsilon$ ( $\forall \varepsilon > 0$ )
1937	Ingham	$\theta = \frac{5}{8} + \varepsilon$
1961	Montgomery	$\theta = \frac{3}{5} + \varepsilon$
1972	Huxley	$\theta = \frac{7}{12} + \varepsilon$
1979	Iwaniec, Jutila	$\theta = \frac{13}{23} + \varepsilon$
1984	Iwaniec, Pintz	$\theta = \frac{1}{2} + \frac{1}{21} + \varepsilon = 0.547\dots + \varepsilon$
1994	Lou and Yeo	$\theta = \frac{7}{13} + \varepsilon = 0.538\dots + \varepsilon$
1998	Baker and Herman	$\theta = 0.535\dots + \varepsilon$

### Exercise for John in his Retirement

[Hardy and Wright, 1979]: There is a prime  $p$  with  $n^2 < p < (n+1)^2$

Note:  $\theta = \frac{1}{2} \Rightarrow \exists p$   $x < p < x + \sqrt{x}$ .  
 $x = n^2 \Rightarrow n^2 < p < n^2 + n < (n+1)^2$ .

Degree of difficulty for the student:

### Other Results on the Distribution of Primes

**Theorem 21 (Bertrand's Postulate)**  $\forall n \in \mathbb{N}, \exists p \in \mathbb{P}$  with  $n \leq p < 2n$ .

**Theorem 22** *There are infinitely many primes of the form  $4n - 1$ .*

*Proof.* Assume there are only a finite number and let  $p$  be the largest. Let

$$N = 2^2 \cdot 3 \cdot \overbrace{5 \cdots p}^n - 1$$

The product  $n = 3 \cdot 5 \cdots p$  contains all the odd primes less than or equal to  $p$  as factors. Since  $N > p$  and  $N = 4n - 1$ , it cannot be prime. No prime less than or equal to  $p$  divides  $N$  (since it would divide 1). Thus all the prime factors of  $N$  must exceed  $p$ .

If  $x = 4m + 1$  and  $y = 4\ell + 1$  then

$$xy = 16m\ell + 4m + 4\ell + 1 = 4(4m\ell + m + \ell) + 1 = 4k + 1$$

If two factors of  $N$  are of the form  $4n + 1$ , so is their product. But  $N$  has the form  $4n - 1$ , so at least one prime factor must be of the form  $p = 4m - 1$ . This contradiction proves the theorem.  $\square$

Can also show there are an infinite number of primes of each of the forms  $4n + 1$ ,  $5n - 1$ ,  $8n - 1$ ,  $8n - 3$  and  $8n + 3$ .

*Note* All numbers of the form  $4n$  or  $4n + 2$  are composite. Every prime  $p \in \mathbb{P}$  is of the form  $4n + 1$  or  $4n + 3$ .

**Theorem (Dirichlet)** *If  $k > 0$  and  $(h, k) = 1$  then  $\forall x > 1$ ,*

$$\sum_{\substack{p \leq x \\ p \equiv h \pmod{k}}} \frac{\log(p)}{p} = \frac{1}{\phi(k)} \log(x) + O(1)$$

**Corollary** *Since  $x \rightarrow \infty \Rightarrow \log(x) \rightarrow \infty$ , there are an infinite number of primes in every arithmetic progression  $nk + h$ ,  $n = 0, 1, 2, 3, \dots$  since  $p = nk + h$  for some  $n \Leftrightarrow p \equiv h \pmod{k}$ .*

**Theorem (Dirichlet)** *Let*

$$\pi_h(x) = \sum_{\substack{p \leq x \\ p \equiv h \pmod{k}}} 1.$$

*Then  $\pi_h(x)$  counts the number of primes in  $nk + h$ ,  $n = 0, 1, 2, 3, \dots$*

$$\pi_h(x) \sim \frac{\pi(x)}{\phi(k)} \sim \frac{1}{\phi(k)} \frac{x}{\log(x)} \text{ as } x \rightarrow \infty$$

**Corollary** *For each  $h \pmod{k}$ ,  $\pi_h(x)$  has the same asymptotic value i.e. the number of primes in each class  $[h]_k$  is asymptotically the same.*

*Note* All attempts to extend this result to more complex subsets of  $\mathbb{N}$  than arithmetic progressions have failed.

1. Are there an infinite number of primes of the form  $p = n^2 + 1$ ?

**Ex** There are an infinite number of composites  $xy = n^2 + 1$ .

2. Are there an infinite number of primes  $p$  such that  $q = 2p + 1$  is also prime? (Sophie Germain primes.)

3. Are there an infinite number of primes  $p$  such that  $q = p + 2$  is also prime? (Twin primes conjecture.)

**Ex** If  $n > 3$  one of  $\{n, n + 2, n + 4\}$  is divisible by 3, and is hence composite. (No triple primes conjecture.)

*Proof:*

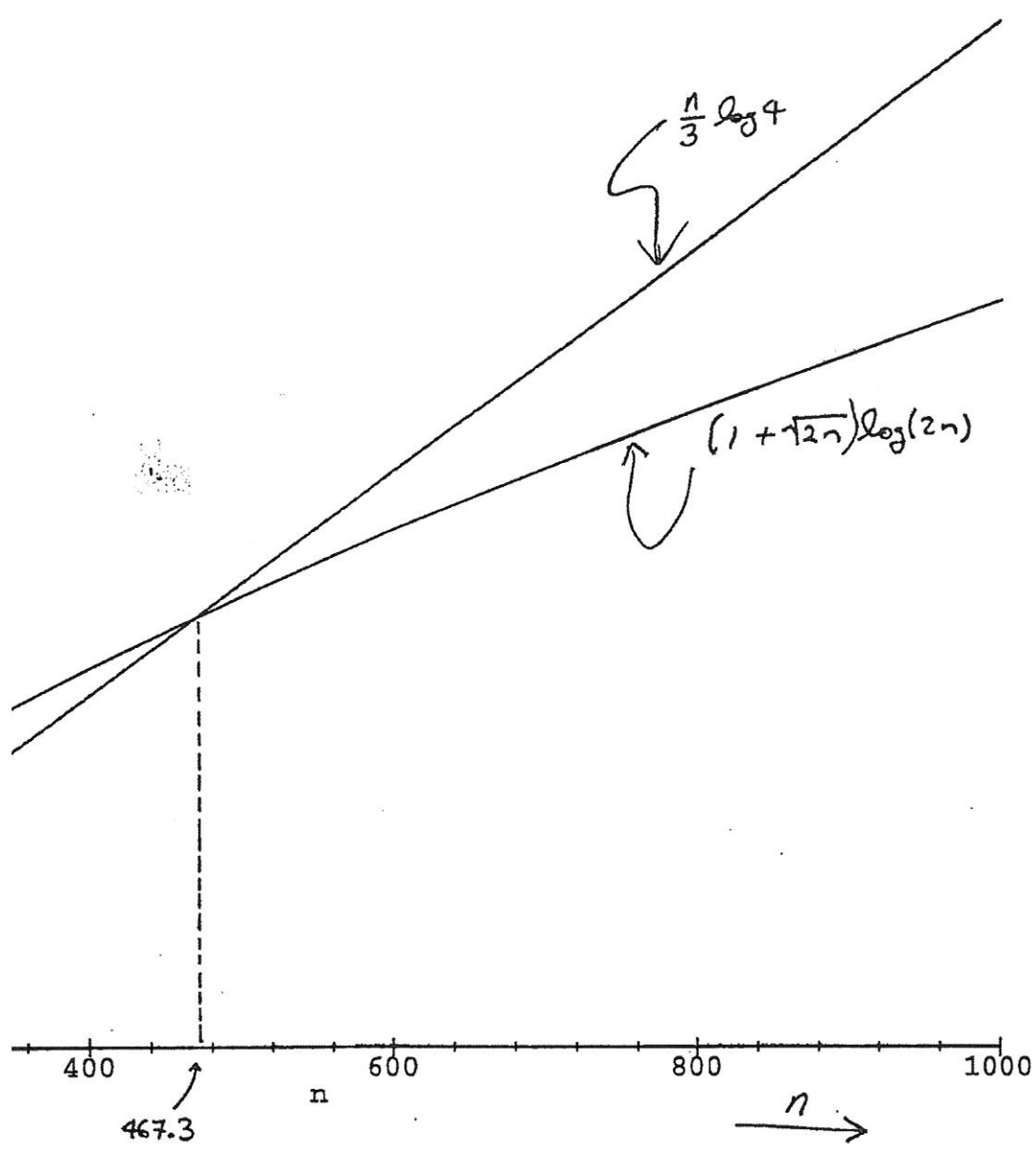
$$n \equiv 0 \pmod{3} \Rightarrow 3 | n$$

$$n \equiv 1 \pmod{3} \Rightarrow n + 2 \equiv 3 \equiv 0 \pmod{3} \Rightarrow 3 | n + 2$$

$$n \equiv 2 \pmod{3} \Rightarrow n + 4 \equiv 6 \equiv 0 \pmod{3} \Rightarrow 3 | n + 4$$

□

4. Find a *quadratic polynomial*  $f(n) = an^2 + bn + c$  with an *infinite number* of prime values.



EASY

