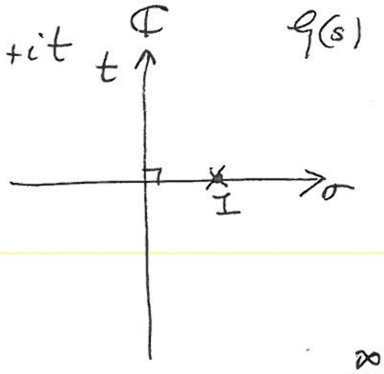
Ch 3 The Riemann Zeta Function

§3.1 Definition $\zeta : \mathbb{C} \setminus \{1\} \longrightarrow \mathbb{C}$

$s = \sigma + it$

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \cdots, \quad \sigma > 1.$$



Proposition 1 The series $\sum_{n=1}^{\infty} \frac{1}{n^s}$ converges absolutely for $\sigma > 1$

Proof

$$\left| \sum_{n=1}^{N} \frac{1}{n^s} \right| \leq \sum_{n=1}^{N} \frac{1}{|n^s|} = \sum_{n=1}^{N} \frac{1}{n^\sigma}$$

modulus

since $|n^s| = |n^{\sigma+it}| = |n^\sigma||n^{it}| = n^\sigma |e^{it\log n}| = n^\sigma$

Now $\displaystyle\int_1^\infty \frac{dx}{x^\sigma} = \lim_{T\to\infty} \int_1^T \frac{dx}{x^\sigma} = \lim_{T\to\infty} \left(\frac{x^{1-\sigma}}{1-\sigma} \Bigg|_{x=1}^{x=T} \right) = \frac{1}{\sigma-1} < \infty$ conv.

so by the integral test, so does $\sum_{n=1}^{\infty} \frac{1}{n^\sigma}$. Thus the series for $\zeta(s)$ converge absolutely for $\sigma > 1$ //

Proposition 2 For $\sigma > 1$ $\quad \zeta(s) = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}$ where the product is over all primes p.

Proof Note that for $|z| < 1$: $1 + z + z^2 + \cdots = \frac{1}{1-z}$, geometric series.

Let $z = \frac{1}{p^s}$. Then $|z| = \frac{1}{p^\sigma} \leq \frac{1}{2^\sigma} < 1$ so

$$\prod_p \left(1 - \frac{1}{p^s}\right)^{-1} = \prod_p \frac{1}{1 - \frac{1}{p^s}} = \prod_p \left(1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \frac{1}{p^{3s}} + \cdots\right)$$

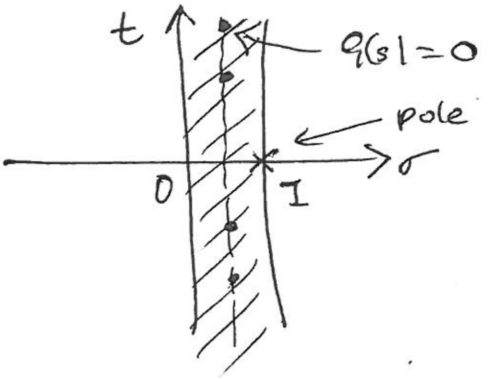
$$= \left(1 + \frac{1}{2^s} + \frac{1}{2^{2s}} + \frac{1}{2^{3s}} + \cdots\right)$$

$$\times \left(1 + \frac{1}{3^s} + \frac{1}{3^{2s}} + \frac{1}{3^{3s}} + \cdots\right)$$

$$\times (1 \ldots)$$

$$= 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{2^{2s}} + \frac{1}{5^s} + \frac{1}{2^s . 3^s} + \cdots = \zeta(s) \text{ //}$$

Note: $\zeta(s)$ is defined by the series and product only for $\sigma > 1$.
But $\exists$ on $\mathbb{C}\setminus\{1\}$. It has a simple pole at $s=1$ with residue 1. We need $\zeta(s)$ for $0 < \sigma \le 1$, the critical strip.



Theorem 3.2 For $\sigma > 0$, $s \neq 1$ we have $\boxed{\zeta(s) = \frac{s}{s-1} - s\int_1^\infty \frac{x - \lfloor x \rfloor}{x^{s+1}}\,dx}$

Proof Let $\sigma > 1$. Use Abel summation with $a_n = 1$, $f(t) = \frac{1}{t^s}$ so.
$x > 1$

$$\sum_{1 < n \le x} \frac{1}{n^s} = \frac{\lfloor x \rfloor}{x^s} - \frac{1}{1^s} + s\int_1^x \frac{\lfloor u \rfloor}{u^{s+1}}\,du$$ and let $x \to \infty$.

$$\zeta(s) = 0 + s\int_1^\infty \frac{\lfloor u \rfloor}{u^{s+1}}\,du$$

$$= s\int_1^\infty \frac{u - (u - \lfloor u \rfloor)}{u^{s+1}}\,du = s\int_1^\infty \frac{du}{u^s} - s\int_1^\infty \frac{u - \lfloor u \rfloor}{u^{s+1}}\,du$$

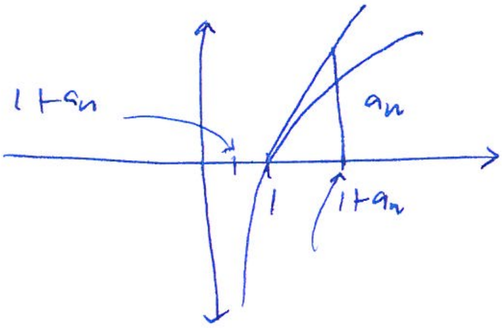
$$= s\left(\frac{u^{1-s}}{1-s}\right)\Bigg|_{u=1}^{u=\infty} - s\int_1^\infty \frac{u - \lfloor u \rfloor}{u^{s+1}}\,du$$

$$= \frac{s}{s-1} - s\int_1^\infty \frac{u - \lfloor u \rfloor}{u^{s+1}}\,du \quad //$$

But each term on the RHS is valid for $\sigma > 0$, so can be used to express $\zeta(s)$ when $0 < \sigma \le 1$. //

Conjecture 3.4 [Riemann Hypothesis: Bernhard Riemann 1856]
$\zeta(s) = 0$ with $s = \sigma + it$ and $\sigma > 0 \Rightarrow \sigma = \frac{1}{2}$. i.e.
All zeros of $\zeta(s)$ in the right half plane are on the critical line $\sigma = \frac{1}{2}$.

<u>Lemma</u>: $|a_n| < \frac{1}{2} \Rightarrow \log(1+a_n) \simeq a_n$.

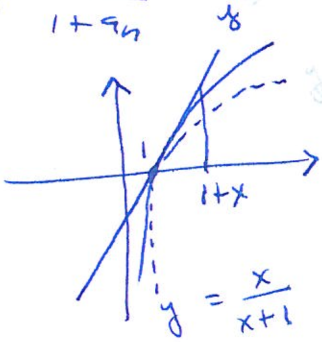


If $a_n > 0$ $\quad \log(1+a_n) < a_n$

If $-\frac{1}{2} < a_n < 0$

$$\Rightarrow \quad \frac{1}{2} < 1+a_n \quad \Rightarrow \quad 2 > \frac{1}{1+a_n} \quad \Rightarrow \quad 2a_n < \frac{a_n}{1+a_n}$$

<u>But</u> $\quad -1 < x \quad \Rightarrow \quad \frac{x}{x+1} \leq \log(1+x) \leq x$

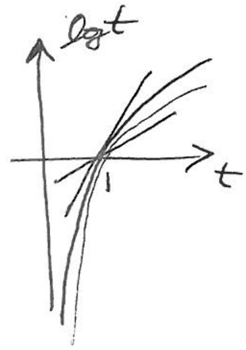


$$\therefore \quad 2a_n < \log(1+a_n) < a_n$$

$$\Rightarrow \quad \log(1+a_n) \simeq a_n.$$

<u>Lemma 17.5</u> (p277) If $a_n \in \mathbb{C}$ $n \geq 1$ & $\sum_{n=1}^{\infty} |a_n| < \infty$ $\boxtimes$ then

$\lim_{N \to \infty} \prod_{n=1}^{N} (1+a_n)$ $\exists$ in $\mathbb{C}$ and is <u>non zero</u> and finite.



<u>Proof</u> Let $a_n \in \mathbb{R}$.

Let $|a_n| < \frac{1}{2}$ for $n > n_0$.

Hence $\log(1+a_n) \simeq a_n$ for $n > n_0$

Thus $\sum_{n=1}^{\infty} \log(1+a_n)$ converges

Let $\sum_{n=1}^{\infty} \log(1+a_n) = L \Rightarrow \prod_{n=1}^{\infty}(1+a_n) = e^{L} \neq 0$.

Now let $a_n \in \mathbb{C}$ and suppose $\prod_{n=1}^{\infty}(1+a_n) = 0$. $\Rightarrow$

$\prod_{n=1}^{\infty} |1+a_n| = |0| = 0$ also.

Let $b_n := |1+a_n| - 1 \in \mathbb{R}$

$a_n =: x_n + iy_n$ so

$$b_n = \sqrt{(1+x_n)^2 + y_n^2} - 1 = \frac{x_n^2 + 2x_n + y_n^2}{1 + \sqrt{(1+x_n)^2 + y_n^2}} \Rightarrow$$

$|b_n| \leq 2|x_n| + |a_n|^2 \leq 3|a_n|^2 = O(|a_n|)$ as $n \to \infty$.

By $\boxtimes$ $\sum_{n=1}^{\infty} |b_n|$ converges.

Since $(b_n)_{n \geq 1}$ is a $\mathbb{R}$ seq., by the first part

$0 = \prod_{n=1}^{\infty} |1+a_n| = \prod_{n=1}^{\infty} (1+b_n) \neq 0$. Hence $\sum_{n=1}^{\infty}(1+a_n) = L \neq 0$.

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§3.4 The zeta zeros

Trivial zeros: $-$ve even integers $\zeta(-2) = \zeta(-4) = \zeta(-6) = \dots = 0$

non-trivial zeros

$\frac{1}{2} + i14.13..$ $\quad$ $\frac{1}{2} - i14.13..$

$\frac{1}{2} + i21.02..$ $\quad$ $\frac{1}{2} - i21.02..$

$\frac{1}{2} + i25.01...$ $\quad$ $\frac{1}{2} - i25.01..$

Note
$$\overline{\zeta(s)} = \overline{\sum_1^\infty \frac{1}{n^s}} = \sum_1^\infty \frac{1}{\overline{n^s}} = \sum_1^\infty \frac{1}{n^{\bar{s}}} = \zeta(\bar{s}) \Rightarrow \zeta(s) = 0 \Leftrightarrow \zeta(\bar{s}) = 0.$$

Theorem 3.6 If $\zeta(s) = 0$, then $\sigma < 1$.

Proof Let $\sigma > 1$ and write
$$\zeta(s) = \prod_p \left(1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \dots\right) = \prod_{n \geq 1}\left(1 + \frac{1}{p_n^s} + \frac{1}{p_n^{2s}} + \dots\right)$$

where $\cdot$ p_n is the n^{th} prime

Let
$$a_n = \frac{1}{p_n^s} + \frac{1}{p_n^{2s}} + \dots = \frac{1}{p_n^s\left(1 - \frac{1}{p_n^s}\right)} \quad \text{so } \zeta(s) = \prod_{n=1}^\infty (1 + a_n)$$

and
$$|a_n| = \frac{1}{|p_n^s|\left|1 - \frac{1}{p_n^s}\right|} \leq \frac{2}{p_n^\sigma} \Rightarrow \sum_{n=1}^\infty |a_n| < \infty.$$

Therefore, by Lemma 17.5, $\sigma > 1 \Rightarrow \zeta(s) \neq 0$.

If $\sigma = 1$ we use a clever trick (known for > 100 years!).

Let $\theta \in \mathbb{R}$. Then
$$0 \leq 2(1 + \cos\theta)^2 = 2(1 + \cos^2\theta + 2\cos\theta)$$
$$= \underbrace{2\cos^2\theta - 1}_{\cos(2\theta)} + 3 + 4\cos\theta$$
$$= 3 + 4\cos\theta + \cos(2\theta)$$

$\boxtimes$

$\therefore$ $0 \leq 3 + 4\cos\theta + \cos(2\theta)$. $\forall \theta \in \mathbb{R}$

now use
$$\log(1 - x) = -x - \frac{1}{2}x^2 - \frac{1}{3}x^3 - \dots \quad |x| < 1,\ x \in \mathbb{C}.$$

and
$$\log \zeta(s) = \log \prod_p \left(1 - \frac{1}{p^s}\right)^{-1} = -\log \prod_p \left(1 - \frac{1}{p^s}\right)$$
$$= -\sum_p \log\left(1 - \frac{1}{p^s}\right) = \sum_p \sum_{n \geq 1} \frac{1}{n p^{sn}}$$

Now if $a>0$, & $s=\sigma+it$

$Re(a^{sn}) = Re(a^{\sigma n+itn}) = a^{\sigma n} Re(a^{itn}) = a^{\sigma n} Re(e^{itn\log a})$

$= a^{\sigma n}(\cos(tn\log a))$, Let $a=\frac{1}{p}>0 \Rightarrow$

$\therefore \; Re\left(\frac{1}{p^{sn}}\right) = \frac{\cos(-tn\log p)}{p^{\sigma n}}$

$= \frac{\cos(tn\log p)}{p^{\sigma n}}$.

So $\underbrace{Re\left(\log(\zeta(s))\right)}_{\substack{\parallel \\ \log|\zeta(s)|}} = \sum_p \sum_{n\geq 1} Re\left(\frac{1}{np^{sn}}\right) = \sum_p \sum_{n\geq 1} \frac{\cos(tn\log p)}{np^{\sigma n}}$

Since $\log z = \log|z| + i\arg z$.

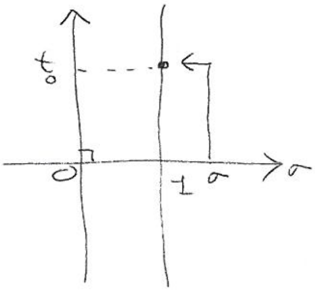
Let $\theta = nt\log p$ in $\boxtimes$ to get

$\frac{3+4\cos(nt\log p)+\cos(2nt\log p)}{np^{\sigma n}} \geq 0$. $\forall n\geq 1$.

Thus $3\log|\zeta(\sigma)| + 4\log|\zeta(\sigma+it)| + \log(|\zeta(\sigma+2it)|) \geq 0$

or (*) $|\zeta(\sigma)|^3|\zeta(\sigma+it)|^4|\zeta(\sigma+2it)| \geq e^0 = 1$ taking $\exp(\cdot)$.

Now suppose $\zeta(1+it_0)=0$ (??)



Let $\sigma \to 1+$

(i) $\zeta(s) = \frac{s}{s-1} - s\int_1^\infty \frac{\{x\}}{x^{s+1}} \Rightarrow (\sigma-1)\zeta(\sigma) = \underset{\substack{\downarrow\\1}}{\sigma} - \underset{\substack{\downarrow\\0}}{(\sigma-1)}\underset{\substack{\downarrow\\1}}{\sigma}\underbrace{\int_1^\infty \frac{\{x\}}{x^{\sigma+1}}dx}_{\text{bdd.}}$

$= O(1)$.

$\Rightarrow |\zeta(\sigma)| = O\left(\frac{1}{\sigma-1}\right)$, $\sigma\to 1+$.

(ii) $\zeta'(1+it_0)\ \exists \Rightarrow \zeta'(1+it_0) \approx \frac{\zeta(\sigma+it_0)-\zeta(1+it_0)}{(\sigma-1)} = \frac{\zeta(\sigma+it_0)}{\sigma-1} \Rightarrow$

$|\zeta(\sigma+it_0)| = O(\sigma-1)$

(iii) $\zeta(s)$ is continuous at $s=1+2it_0 \Rightarrow |\zeta(\sigma+2it_0)| = O(1)$, $\sigma\to 1+$.

(*) $\Rightarrow$. $1 \leq |\zeta(\sigma)|^3 |\zeta(\sigma+it_0)|^4 |\zeta(\sigma+2it_0)| = O\left(\frac{1}{(\sigma-1)^3}(\sigma-1)^4\right) = O(\sigma-1) \to 0$

which is a contradiction. //

Note: This was one of the essential & difficult steps needed to prove the Prime number theorem about 1900, after 50 years of work, at least.

73.5 Euler's Sum

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \dots = \zeta(2) = \frac{\pi^2}{6}$$

(- also proof using Fourier series).

Proof: [Apostol] Define

$$I = \int_0^1\int_0^1 \frac{1}{1-xy}\,dx\,dy = \lim_{\varepsilon \to 0+} \int_0^{1-\varepsilon}\int_0^{1-\varepsilon} \frac{1}{1-xy}\,dx\,dy$$

As a geometric series

$$\frac{1}{1-xy} = \sum_{n=0}^{\infty} (xy)^n.$$ Assume all steps can be made rigorous

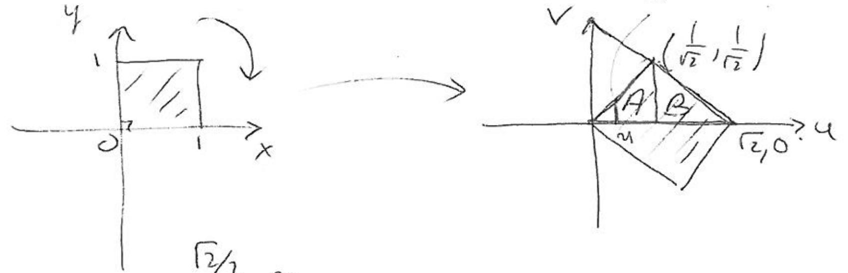
$$I = \int_0^1\int_0^1 \sum_{n=0}^{\infty}(xy)^n\,dx\,dy = \sum_{n=0}^{\infty}\left(\int_0^1 x^n\,dx\right)\cdot\left(\int_0^1 y^n\,dy\right) = \sum_{n\geq 0} \frac{1}{n+1}\cdot\frac{1}{n+1}$$

$$= \sum_{n=1}^{\infty} \frac{1}{n^2} = \zeta(2).$$

Now evaluate I making a change of variables

$$\left.\begin{array}{l} u = \dfrac{y+x}{\sqrt{2}} \\ v = \dfrac{y-x}{\sqrt{2}} \end{array}\right\} \Rightarrow \begin{array}{l} x = \dfrac{u-v}{\sqrt{2}} \\ y = \dfrac{u+v}{\sqrt{2}} \end{array}$$ and $1 - xy = 1 - \dfrac{u^2-v^2}{2} \Rightarrow$

$$\frac{1}{1-xy} = \frac{2}{2-u^2+v^2} = f(u,v).$$



Symmetry of the fcn & domain

$$\Rightarrow I = 4\iint_A + 4\iint_B = 4I_A + 4I_B$$

$$\frac{1}{2-u^2+v^2}$$

$$I_A = \int_0^{\sqrt{2}/2}\left(\int_0^u \frac{dv}{(2-u^2)+v^2}\right)du$$

$$= \int_0^{\sqrt{2}/2} \frac{1}{\sqrt{2-u^2}} \arctan\left(\frac{u}{\sqrt{2-u^2}}\right)du$$

Let $u = \sqrt{2}\sin\theta$
$0 \leq u \leq \sqrt{2}/2 \iff 0 \leq \theta \leq \frac{\pi}{6}$.

$$= \int_0^{\pi/6} \frac{1}{\sqrt{2}\cos\theta} \arctan\left(\frac{\sqrt{2}\sin\theta}{\sqrt{2}\cos\theta}\right)\sqrt{2}\cos\theta\,d\theta$$

$$= \int_0^{\pi/6} \theta\,d\theta = \frac{1}{2}\left(\frac{\pi}{6}\right)^2 = \cancel{\frac{1}{3}\frac{\pi}{6}}.$$

Since $I_B = \left(\frac{\pi}{6}\right)^2$ so

$$\zeta(2) = I = \frac{4}{2}\frac{\pi^2}{6^2} + \frac{4\pi^2}{6^2} = \frac{\pi^2}{6^2}\overset{2+4}{\left(\cancel{\tfrac{1}{3}} + \cancel{\tfrac{2}{3}}\right)}$$

$$= \frac{\pi^2}{6}. \quad //$$