Notation

$a|b$ means $a, b \in \mathbb{Z}$ and $ac = b$ for some $c \in \mathbb{Z}$.

p, q prime numbers i.e. $d|p \Rightarrow d = 1$ or $d = p$. $d, p \in \mathbb{N}$.

$\lfloor x \rfloor = \max\{n \in \mathbb{Z} : n \leq x\}$.

$\{x\} = x - \lfloor x \rfloor$

$p^a \| b$ means $p^a | b$ & $p^{a+1} \nmid b$ Ex $2^2 \| 12$.

(a, b) is the greatest common divisor of a and b

$\log x$ is the natural logarithm $\log_e(x)$ or $\ln(x)$.

Landau's notation: $f(x) = O(g(x))$ as $x \to \infty$ if $\exists M, x_0$ so.

$$|f(x)| \leq M g(x) \ \forall x \geq x_0.$$

also use $f(x) \ll g(x)$.

$f(x) \sim g(x)$ $\lim_{x \to \infty} \frac{f(x)}{g(x)} = 1$.

$f(x) \asymp g(x)$ $f(x) \ll g(x)$ and $g(x) \ll f(x)$

Ex $x^2 + x \asymp x^2$ since $x^2 \leq x^2 + x$ & $x^2 + x \leq 2x^2$, $x \geq 1$.

$\sum_{n \leq x} a_n$ means $1 \leq n \in \mathbb{N}$, $x \in \mathbb{R}^+$.
is $\{a_1 + a_2 + a_3 + \dots + a_N : 1 \leq n \leq x\}$.

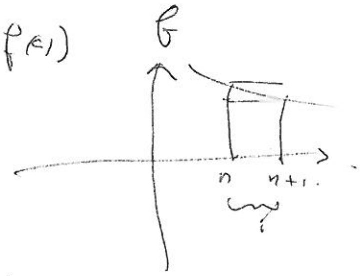
So $\sum_{n \leq 3.5} n^2 = 1^2 + 2^2 + 3^2 = \sum_{n \leq 3.9} n^2$

Prop 1.1 $a,b \in \mathbb{N}$

$f:[a,b] \to \mathbb{R}$ decreasing:

$\exists\,\theta$ $\quad 0 \le \theta \le 1$ with

$$\sum_{a<n\le b} f(n) = \int_a^b f(t)\,dt + \theta\big(f(b)-f(a)\big)$$



Proof. note $\forall n$, $a \le n \le b-1$: $\quad f(n+1) \le \int_n^{n+1} f(t)\,dt \le f(n)$

$$\therefore \int_a^b f(x)\,dx = \sum_{n=a}^{n=b-1} \int_n^{n+1} f(x)\,dx$$

$$\ge \sum_{n=a}^{n=b-1} f(n+1) \left(= \sum_{a<n\le b} f(n)\right)$$

$$\therefore\ 0 \le \int_a^b f(x)\,dx - \sum_{a<n\le b} f(n) < \sum_{n=a}^{n=b-1} f(n) - \sum_{n=a}^{n=b-1} f(n+1)$$

$$= f(a) - f(b)$$

$$\therefore \int_a^b f(x)\,dx = \sum_{a<n\le b} f(n) + \theta\big(f(a) - f(b)\big).$$

Application

$$\log(1) = 0 \Rightarrow \log(n!) = \sum_{j=2}^{n} \log(j) = \sum_{1<j\le n} \log(j)$$

$$= \int_1^n \log t\,dt + \theta\big(\log(n) - \log(1)\big)$$

$$= (t\log t - t)\Big|_1^n + O(\log n). \quad \text{as } n \to \infty$$

$$= n\log n - n + O(1) + O(\log n)$$

$$= n\log n - n + O(\log n)$$

$$\therefore\ n! \sim e^{n\log n - n + O(\log n)} = \left(\frac{n}{e}\right)^n e^{O(\log n)}$$

Laplace's Method of Steepest descent

This is an application of $\int_{-\infty}^{+\infty} e^{-\pi x^2} dx = 1.$

to give an asymptotic estimate for $\int_a^b e^{t f(x)} g(x)\, dx$

for $t \to \infty$. There are conditions on f and g given below.

An application is Stirling's approx. to $n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$, $n \to \infty$

<u>Lemma</u> (Taylor's formula with integral remainder)

Let $f \in C^{n+1}[0,x]$. Then.

(n) $$\boxed{f(x) = \sum_{j=0}^{n} f^{(j)}(0)\, \frac{x^j}{j!} + \int_0^x f^{(n+1)}(t)\, \frac{(x-t)^n}{n!}\, dt}.$$

For $n=1$ $\square \Leftrightarrow$

(1) $$f(x) = f(0) + \int_0^x f'(t)\, dt, \quad f \in C^1[0,x]$$

which is the fundamental theorem of calculus.

(2) $$f(x) = f(0) + f'(0)x + \int_0^x f''(t)(x-t)\, dt, \quad f \in C^2[0,x].$$

To get (2) from (1) we need to show

$$\int_0^x f'(t)\, dt = f'(0) + \int_0^x f''(t)(x-t)\, dt.$$

Integrate by parts with $u = f'(t)$, $\begin{array}{l} v = t-x \\ dv = dt \end{array}$ so.

$$\int_0^x f'(t)dt = \int_{t=0}^{t=x} u\, dv = uv\Big|_{t=0}^{t=x} - \int_{t=0}^{t=x} v\, du$$

$$= f'(t)(t-x)\Big|_0^x - \int_0^x f''(t)(t-x)dt$$

$$= f'(0)\, x - \int_0^x f''(t)(t-x)dt \quad \text{i.e. (2).}$$

So the proof is by induction. We need only show $(n-1) \Rightarrow (n)$.

(n−1) has "remainder", and

$$(n) \quad \int_0^x f^{(n)}(t)\frac{(x-t)^{n-1}}{(n-1)!}\,dt = f^{(n)}(0)\frac{x^n}{n!} + \int_0^x f^{(n+1)}(t)\frac{(x-t)^n}{n!}\,dt$$

for $f \in C^{n+1}[0,x]$ can be verified using.

$$u = f^{(n)}(t) \qquad v = -\frac{(x-t)^n}{n!}$$

$$du = f^{(n+1)}(t)\,dt \qquad dv = \frac{(x-t)^{n-1}}{(n-1)!}\,dt.$$

<u>Theorem</u>(L) Let $f \in C^3[a,b]$, $g \in C^1[a,b]$

$f'(a) = 0$, $f''(x) < 0$ on $[a,b]$, $g(a) \neq 0$.

Then

(a)
$$\boxed{\int_a^b g(x)\,e^{t f(x)}\,dx \sim \left(\frac{1}{2}\right) e^{t f(a)}\, g(a)\sqrt{\frac{2\pi}{t(-f''(a))}}, \quad t \to \infty}$$

where $\theta(x) \sim \varphi(x)$ as $x \to \infty$ means $\lim_{x\to\infty} \frac{\theta(x)}{\varphi(x)} = 1$.

(a_∞) If $\int_a^\infty |g| < \infty$ then the estimate (a) holds with

$b = \infty$

(c) If $a < c < b$, $f'(c) = 0$, $f''(x) \leq 0$ on $[a,b]$, $g(c) \neq 0$

then $\square$ is true with $\left(\frac{1}{2}\right)$ replaced by 1.

(c_∞) If $\int_{-\infty}^{+\infty} |g| < \infty$ then ....

<u>Application</u>

$$n! = \int_0^\infty e^{-x} x^n\,dx \qquad n = 0, 1, 2, \ldots$$

let $nu = x$, $n\,du = dx$ so.

$$n! = \ldots = n^{n+1}\int_0^\infty e^{n(\log x - x)}\,dx.$$

let $f(x) = \log x - x \implies f'(x) = \frac{1}{x} - 1$ so $f'(1) = 0$ and

$f''(x) = -\frac{1}{x^2} < 0$ for $1 \leq x < \infty$.

By Thm (L) (a_∞) with $a = 1$

$$n! \sim n^{n+1}\sqrt{\frac{2\pi}{n}}\, e^{n f(1)} = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \quad //$$

(Thm 1.14)

Proof of (L)

(a) Change variable so we can assume $a=0$. Let $\delta>0$.

For $x \in [a,\delta]$, $\delta \in [a,b]$, by the Lemma.

$$f(x) = f(0) + \frac{1}{2} f''(0) x^2 + O(\delta^3)$$

$$g(x) = g(0)(1+O(\delta))$$

Thus

$$I = \int_0^\delta e^{tf(x)} g(x)\,dx = e^{tf(0)} g(0) \int_0^\delta e^{\frac{1}{2} t f''(0) x^2} e^{O(t\delta^3)} (1+O(\delta))\,dt$$

Choose $\delta = \delta_t$ so that $t\delta^3 \to 0$ as $t \to \infty$ (e.g. $\delta = \frac{1}{t}$) then

$$e^{O(t\delta^3)}(1+O(\delta)) \to 1 \quad \text{as } t\to\infty \quad \text{uniformly for } x \in [0,\delta]$$

Hence

$$I \sim e^{tf(0)} g(0) \int_0^\delta e^{\frac{1}{2} t f''(0) x^2}\,dx.$$

$$u = x\sqrt{\frac{-tf''(0)}{2\pi}} \Rightarrow \quad = \sqrt{\frac{2\pi}{-tf''(0)}} \int_0^{\delta\sqrt{\frac{-tf''(0)}{2\pi}}} e^{-\pi u^2}\,du$$

If we choose $\delta = \delta_t$ so $t\delta^2 \to \infty$ as $t\to\infty$, the last integral has value $\frac{1}{2}$. E.g. $\boxed{\delta = t^{-2/5}}$ works for $t\delta^3 \to 0$ and $t\delta^2 \to \infty$.

Therefore, for this δ, as $t\to\infty$

$$\text{(M)} \qquad \int_0^\delta e^{tf(x)} g(x)\,dx \sim \frac{1}{2}\sqrt{\frac{2\pi}{-tf''(0)}}\, e^{tf(0)} g(0)$$

---

for $x \in [\delta, b]$ $\quad f''(x) < 0 \Rightarrow \quad f'(x) - f'(a) = f''(\xi)(x-a) < 0.$

$\Rightarrow \quad f'(x) < 0$ so f is <u>decreasing</u>. Thus with $\delta = t^{-2/5}$,

$$e^{tf(x)} \le e^{tf(\delta)} = e^{tf(0)} e^{\frac{1}{2} f''(0)\delta^2} e^{O(t\delta^3)} \quad \text{and so.}$$

$$\text{(R)} \qquad \left| \int_\delta^b e^{tf(x)} g(x)\,dx \right| \leq e^{tf(0)} e^{\frac{1}{2} f''(0) t^{1/5}} \int_a^b |g(x)|\,dx.$$

which is smaller asymptotically than the RHS of (M) since $f''(0) < 0$.

Then (M) + (R) $\Rightarrow$ (a) //.

Abel Summation

$f : [y, x] \to \mathbb{C}$ contin deriv

(a_n) in $\mathbb{C}$

$A(x) = \sum_{n \le x} a_n$.

$y, x \in \mathbb{N}$.

$$\sum_{y<n\le x} a_n f(n) = \sum_{n=y+1}^{x} a_n f(n) = \sum_{n=y+1}^{x} (A(n) - A(n-1)) f(n).$$

$$= \sum_{n=y+1}^{x} A(n) f(n) - \sum_{n=y}^{x-1} A(n) f(n+1).$$

$$= \sum_{n=y+1}^{x-1} A(n)(f(n) - f(n+1)) + A(x) f(x) - A(y) f(y+1)$$

$$= - \int_{y+1}^{x} A(t) f'(t) + A(x) f(x) - A(y) f(y)$$

$$+ A(y)(f(y) - f(y+1))$$

$$- \int_{y}^{y+1} A(t) f'(t)$$

$$\Rightarrow \boxed{\sum_{y<n\le x} a_n f(n) = A(x) f(x) - A(y) f(y) - \int_{y}^{x} A(t) f'(t)\, dt.}$$

$y \in \mathbb{N}$ $x \notin \mathbb{N}$ let $N = \lfloor x \rfloor$

$$\sum_{y<n\le x} a_n f(n) = \sum_{y<n\le N} a_n f(n) = A(N) f(N) - A(y) f(y) - \int_{y}^{N} A(t) f'(t) dt$$

$$= A(x) f(x) - A(y) f(y) + A(N)(f(N) - f(x)) - \int_{y}^{N} A(t) f'(t) dt$$

$$\text{also} = A(x) f(x) - A(y) f(y) - \int_{y}^{x} A(t) f'(t) dt$$

<u>Def</u> <u>Euler's Constant</u> $\gamma := 1 - \int_1^\infty \frac{t - \lfloor t \rfloor}{t^2}\,dt = 0.5772156690\ldots$

<u>Unsolved</u>: does $\gamma \notin \mathbb{Q}$?

<u>Theorem 1.7</u> $\sum_{1 \le n \le x} \frac{1}{n} = \log x + \gamma + O\left(\frac{1}{x}\right)$ as $x \to \infty$.

<u>Proof</u>

Let $a_n = 1$ and $f(t) = \frac{1}{t}$ in <u>Prop 1.4</u>. Then

$$A(x) = \sum_{n \le x} 1 = \lfloor x \rfloor \quad \text{and}$$

$$f'(t) = -\frac{1}{t^2}, \quad t > 0.$$

Hence.
$$\sum_{1 \le n \le x} \frac{1}{n} = \sum_{1 \le n \le x} a_n f(n) = A(x)f(x) - A(1)f(1) - \int_1^x A(t) f'(t)\,dt$$

$$= \frac{\lfloor x \rfloor}{x} + \int_1^x \frac{\lfloor t \rfloor}{t^2}\,dt - 1$$

$$= \frac{x - (x - \lfloor x \rfloor)}{x} + \int_1^x \frac{t - (t - \lfloor t \rfloor)}{t^2}\,dt - 1$$

$$= 1 + O\left(\frac{1}{x}\right) + \int_1^x \frac{1}{t}\,dt + \int_1^\infty \frac{\lfloor t \rfloor - t}{t^2}\,dt - \int_x^\infty \frac{\lfloor t \rfloor - t}{t^2}\,dt - 1$$

$$= \log x + \left(1 - \int_1^\infty \frac{t - \lfloor t \rfloor}{t^2}\,dt\right) + O\left(\frac{1}{x}\right) + O\left(\frac{1}{x}\right) - 1$$

$$\Rightarrow \sum_{1 \le n \le x} \frac{1}{n} = \log x + \gamma + O\left(\frac{1}{x}\right). \quad //$$

Ex (Problem 1.1) $\alpha \geq 0 \Rightarrow \sum_{1 \le n \le x} n^\alpha = \frac{x^{\alpha+1}}{\alpha+1} + O(x^\alpha)$ as $x \to \infty$.

<u>Ex</u> (Prob 1.3) $\alpha > 0$ $\sum_{n \le x} n^\alpha \log n = \frac{x^{\alpha+1}}{\alpha+1}\left(\log x - \frac{1}{\alpha+1}\right) - O(x^\alpha \log x)$

<u>Ex</u>. $\sum_{n \le x} \frac{\log n}{n^2} = ?$

Theorem 1.12 (Chinese Remainder Theorem)

Let $m_1, \ldots, m_k \in \mathbb{N}$, $(m_i, m_j) = 1$ $i \neq j$, given $a_1, \ldots, a_k \in \mathbb{Z}$. Then $\exists a \in \mathbb{Z}$ so

$a \equiv a_i \bmod m_i$ for $1 \leq i \leq k$ and $a \in [b, b+M) \Rightarrow a$ is unique $\forall b$ given

$M = \prod_{i=1}^{k} m_i$

Ex $(3,4) = 1$ $m_1 = 3$, $m_2 = 4$

$a_1 = 2$, $a_2 = 1$ & $5 \equiv 2 \bmod 3$,

$5 \equiv 1 \bmod 4$.

Proof Let

$$M := \prod_{i=1}^{k} m_i$$

$$M_i := M/m_i$$

then $(m_i, M_i) = 1$ $\forall i = 1, \ldots, k$. so $M_i^{-1} \bmod m_i$ $\exists$. let

$n_i M_i \equiv 1 \bmod m_i$ and let

$$a := \sum_{i=1}^{k} a_i n_i M_i$$

Let $l \in \{1, \ldots, k\}$ so $m_l | M_j$ for $j \neq l$ & mod m_l we get

$$a \equiv a_l n_l M_l \bmod m_l \equiv a_l \bmod m_l$$

Let $a, a' \equiv a_l \bmod m_l$ for $1 \leq l \leq k$

& $b \leq a, a' < b + M$

$\Rightarrow$ $a - a' \equiv 0 \bmod m_l$ $\Rightarrow$ $m_l | a - a'$ $\forall l$ $\Rightarrow$ $M | a - a'$

$\Rightarrow$ $a = a'$ or $M \leq |a - a'| < M$, which is false. Thus $a = a'$. //

Ex find $a \in [200, 200 + 3.5.7)$ so. $a \equiv 1 \bmod 3$,

$a \equiv 2 \bmod 5$,

$a \equiv 4 \bmod 7$.

§1.8 Density of a set of integers

$\lfloor \frac{N}{k} \rfloor k \le N$

& $qk \le N \Rightarrow q \le \frac{N}{k}$

$\Rightarrow q \le \lfloor \frac{N}{k} \rfloor$.

Let $A \subset \mathbb{N}$. Then we say A has *density* δ if

$$\lim_{N\to\infty} \frac{1}{N} \sum_{\substack{n\le N \\ n\in A}} 1 = \delta.$$

i.e. the limit exists (and it will then have value δ: $0 \le \delta \le 1$.

Ex $A_k = \{n \in \mathbb{N} \mid k | n\}$ given $k \in \mathbb{N}$.

Then $\displaystyle\sum_{\substack{n\le N \\ k|n}} 1 = \lfloor \frac{N}{k} \rfloor$

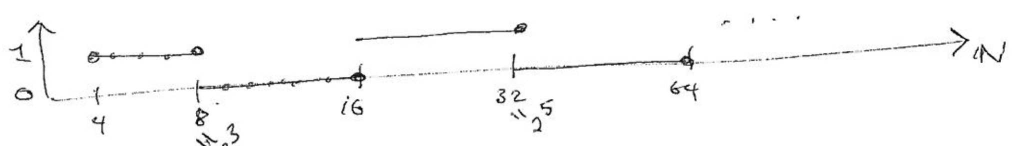
since $N = \lfloor \frac{N}{k} \rfloor k + r$ and $0 \le r < k$ so there are $\lfloor \frac{N}{k} \rfloor$ multiples of k up to N

Hence
$$\delta = \lim_{N\to\infty} \frac{\lfloor \frac{N}{k} \rfloor}{N} = \lim_{N\to\infty} \frac{\frac{N}{k} - \{\frac{N}{k}\}}{N} = \lim_{N\to\infty} \frac{1}{k} - \lim_{n\to\infty} \frac{\{\frac{N}{k}\}}{N}$$
$$= 1/k \quad //$$

Ex. $0 \le b < a$ and

$A = \{n \in \mathbb{N} : n \equiv b \pmod a\} \Rightarrow \dots \quad \delta = \frac{1}{a}$

Ex.



$$f(n) = \begin{cases} 1 & 2^{2m} < n \le 2^{2m+1} \\ 0 & 2^{2m+1} < n \le 2^{2m+2} \end{cases}$$

Then claim $\displaystyle\lim_{N\to\infty} \frac{1}{N} \sum_{\substack{n\le N \\ n\in A}} 1$ does not $\exists$ so A does not have density:

If $N_1 = 2^{2m+1}$ (odd), $\displaystyle A(N_1) = \sum_{0\le n\le m} (2^{2n+1} - 2^{2n}) = \sum_{0\le n\le m} 2^{2n} = \frac{4^{m+1}-1}{3}$.

& $N_2 = 2^{2m+2}$, $A(N_2) = \dfrac{4^{m+1}-1}{3}$ also.

But $\displaystyle\lim_{m\to\infty} \frac{1}{N_1}\left(\frac{4^{m+1}-1}{3}\right) = \lim_{m\to\infty} \frac{4^{m+1}-1}{2^{2m+1}\cdot 3} = \frac{2}{3}$ $\neq$

& $\displaystyle\lim_{m\to\infty} \frac{1}{N_2}\left(\frac{4^{m+1}-1}{3}\right) = \lim_{m\to\infty} \frac{4^{m+1}-1}{2^{2m+2}\cdot 3} = \frac{1}{3}$

Hence $\displaystyle\lim_{N\to\infty} \frac{A(N)}{N} \nexists$ //

Ex: $\mathbb{P} \subset \mathbb{N}$ has $\delta = 0$ (later) $\quad S(N) = \displaystyle\sum_{n^2 \le N} 1 = \lfloor \sqrt{N} \rfloor$

Sqs $\subset \mathbb{N}$ has dens 0.