Ch 6

Global behaviour of Arithmetic functions

$f : \mathbb{N} \to \mathbb{C}$. Let $\omega(n) = \#\{p : p \mid n\}$, Ex $\omega(12) = 2$.

Ex. $\sum_{n \le x} \omega(n) \sim x \log\log x \Rightarrow$ "average" value of $\omega(n)$ is $\log\log n$.

i.e. that is its "global behaviour". Similarly average of $d(n)$ is $\log n$.

Defn f is multiplicative if $\forall\, a,b \in \mathbb{N}$ with $(a,b) = 1$, $f(ab) = f(a)\cdot f(b)$.
f is completely multiplicative if $\forall\, a,b \in \mathbb{N}$, $f(ab) = f(a)\cdot f(b)$.

Ex. $d(n)$ is multiplicative
$f(n) = n^{\alpha}$, $\alpha \in \mathbb{R}$ is completely multiplicative
If $(a,b) = 1$ $\omega(ab) = \omega(a) + \omega(b)$, say ω is additive.

If. $f(n)$ is arithmetic define a Dirichlet series, or generating fun' for $f(n)$ by

$$\varphi_f(s) = \varphi(s) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s}$$

Adopt the (easily removed) restriction $s \in \mathbb{R}$, so $\varphi(s)$ has a $\mathbb{R}$ series representation.

Ex If $f(n) = 2^n$ $\varphi_f(s)$ diverges $\forall\, \mathbb{R}\ s$.
If $f(n) = n$ $\varphi_f(s)$ converge absolutely $\forall\, s > 2$.
If. $f(n) = 1$ $\varphi(s)$ converges for $s > 1$ and diverges for $s \le 1$.

Defn Abscissa of convergence $\alpha_c \in \mathbb{R}$: $\varphi_f(s)$ conv. $s > \alpha_c$, div $s < \alpha_c$.
Abscissa of absolute convergence $\alpha_a \in \mathbb{R}$ conv abs $s > \alpha_a$, div abs $s < \alpha_a$

$$\sum_{n=1}^{\infty} \frac{|f(n)|}{n^s} = \sum_{n=1}^{\infty} \left|\frac{f(n)}{n^s}\right| \text{ here.}$$

so. $\alpha_c \le \alpha_a$

Ex. $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^s}$ $\alpha_c = 0$, $\alpha_a = 1$.

Dirichlet invented these series in 1st half of 19 c used them to show each A.P $a + nb$, $(a,b) = 1$ has an ∞ # of prime values $p_j = a + jb$.

Let
$$I_a = \left\{ s \in \mathbb{R} : \sum_{n=1}^{\infty} \frac{|f(n)|}{n^s} \text{ conv.} \right\}$$
$$J_a = \left\{ s \in \mathbb{R} : \sum_{n=1}^{\infty} \frac{|f(n)|}{n^s} \text{ div.} \right\}.$$
so $\mathbb{R} = I_a \sqcup J_a$.

Theorem 6.1 If $I_a \neq \phi \neq J_a \Rightarrow \alpha_a \in \mathbb{R}$ exists.

Proof. Let $s_1 \in I_a$ and $s_2 \in J_a$. Then $s_2 < s_1$

Let $b = \text{lub } J_a$. Then $b < s \Rightarrow s \in I_a$
$s < b \Rightarrow \exists\ s_1$ so $s < s_1 < b$ and $s_1 \in J_a$.

so $s \in J_a$ $\therefore$ Thus $b = \alpha_a$ //

Ex $\quad \zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$ has $\alpha_a = 1$

$\zeta_\varphi(s) = \sum_{n=1}^{\infty} \frac{\varphi(n)}{n^s}$ where $\varphi(n) = n \prod_{p|n} (1 - \frac{1}{p})$ is Euler's fun.

has $\alpha_a = 2$: $\quad \varphi(n) \leq n \Rightarrow \alpha_a \leq 2$

and $\varphi(p) = p - 1 \Rightarrow \sum_{n=1}^{\infty} \frac{\varphi(n)}{n^s} > \sum_p \frac{\varphi(p)}{p^s} = \sum_p \frac{p-1}{p^s} \geq \frac{1}{2} \sum_p \frac{p}{p^s}$

$\therefore \alpha_a = 2.$ $\qquad = \frac{1}{2} \sum \frac{1}{p^{s-1}} \to \infty\ s \leq 2.$

Thm 6.5 Let two Dirichlet series have the same abscissa of absolute convergence α_a and agree on a sequence $s_i \to \infty$. Then the corresponding arithmetic functs are equal $\forall n \in \mathbb{N}$.

Proof Let $F(s) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s}$ & $G(s) = \sum_{n=1}^{\infty} \frac{g(n)}{n^s}$.

Subtract and set $H(s) := F(s) - G(s) = \sum_{n=1}^{\infty} \frac{h(n)}{n^s}$

where $h(n) = f(n) - g(n)$. Then $H(s)$ conv. abs for $s > \alpha_a$ &

$H(s_i) = 0, \quad i \in \mathbb{N}.$

(??)

Assume $\exists\, n>0$ so. $h(n) \neq 0$ and let $n_0 \in \mathbb{N}$ be the smallest positive integer with $h(n_0) \neq 0$. Then for $s > \alpha_a$.

$$H(s) = \sum_{n=1}^{\infty} \frac{h(n)}{n^s} = \sum_{n=n_0}^{\infty} \frac{h(n)}{n^s} = \frac{h(n_0)}{n_0^s} + \sum_{n=n_0+1}^{\infty} \frac{h(n)}{n^s}$$

$$\therefore\quad h(n_0) = n_0^s H(s) - n_0^s \sum_{n=n_0+1}^{\infty} \frac{h(n)}{n^s}.$$

Because $H(s_k) = 0$ we get

$$h(n_0) = - n_0^{s_k} \sum_{n=n_0+1}^{\infty} \frac{h(n)}{n^{s_k}}.$$

Now let $c > \alpha_a$ and $s_k > c$, so.

$$|h(n_0)| \le n_0^{s_k} \sum_{n=n_0+1}^{\infty} \frac{|h(n)|}{n^{s_k}} = n_0^{s_k} \sum_{n=n_0+1}^{\infty} \frac{|h(n)|}{n^{c+(s_k-c)}}$$

$$\le \frac{n_0^{s_k}}{(n_0+1)^{s_k-c}} \sum_{n=n_0+1}^{\infty} \frac{|h(n)|}{n^c}$$

$$= (n_0+1)^c \left(\frac{n_0}{n_0+1}\right)^{s_k} \sum_{n=n_0+1}^{\infty} \frac{|h(n)|}{n^c}.$$

But as $s_k \to \infty$ $\left(\frac{n_0}{n_0+1}\right)^{s_k} \to 0$ and $c > \alpha_a \Rightarrow \sum_{n=n_0+1}^{\infty} \frac{|h(n)|}{n^c} \le B$

so. $|h(n_0)| \le 0 \Rightarrow h(n_0) = 0$ (!!).

Therefore $\forall n \in \mathbb{N}$, $h(n) = 0$. // ie values are det on any (s_k) $s_k \to \infty$. eg

$\zeta(2) = \frac{\pi^2}{6}$
$\zeta(4) = \frac{\pi^4}{90}$

(76.3) Multiplicative Functions and Dirichlet Series

If $f(n)$ is multiplicative, then $n = p_1^{\alpha_1} \cdots p_m^{\alpha_m} \Rightarrow$

$$f(n) = f(p_1^{\alpha_1}) \cdots f(p_m^{\alpha_m}) \quad \text{so.}$$

$$\frac{f(n)}{n^s} = \prod_{i=1}^{m} \frac{f(p_i^{\alpha_i})}{p_i^{\alpha_i s}} \Rightarrow \sum_{n=1}^{\infty} \frac{f(n)}{n^s} = \prod_p \left(1 + \frac{f(p)}{p^s} + \frac{f(p^2)}{p^{2s}} + \cdots\right)$$

Ex. <u>Euler's function</u> $\varphi(p^a) = p^a - p^{a-1} = p^{a-1}(p-1) = p^a(1-\frac{1}{p})$

Def $\varphi(n) = \#\{j \le n \mid (n,j)=1\}$

so
$$\frac{f(p_i^{\alpha_i})}{p_i^{\alpha_i s}} = \frac{p_i^{\alpha_i - 1}(p_i - 1)}{p_i^{\alpha_i s}} \quad \text{so.}$$

$$\sum_{n=1}^{\infty} \frac{\varphi(n)}{n^s} = \prod_p \left(1 + \frac{(p-1)}{p^s} + \frac{p(p-1)}{p^{2s}} + \frac{p^2(p-1)}{p^{3s}} + \dots\right) = \dots$$

(see later) $= \dfrac{\zeta(s-1)}{\zeta(s)}$, $Re(s) > 2$.

Ex <u>Möbius function</u>
$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} = \prod_p \left(1 + \frac{\mu(p)}{p^s} + \frac{\mu(p^2)}{p^{2s}} + \dots\right)$$
$$= \prod_p \left(1 - \frac{1}{p^s}\right) = \frac{1}{\zeta(s)} \quad \text{for } \sigma > 1.$$

Ex. $f: \mathbb{N} \to \mathbb{C}$ <u>completely multiplicative</u> (so $\forall a,b$ $f(ab) = f(a)f(b)$).

$$\frac{f(p_i^{\alpha_i})}{p_i^{\alpha_i s}} = \frac{(f(p_i))^{\alpha_i}}{p_i^{\alpha_i s}} = \left(\frac{f(p_i)}{p_i^s}\right)^{\alpha_i} = r_{p_i}^{\alpha_i} \quad \text{Assume } |r_{p_i}| < 1$$

$$\sum_{n=1}^{\infty} \frac{f(n)}{n^s} = \prod_p \left(1 + r_p + r_p^2 + r_p^3 + \dots\right)$$
$$= \prod_p \left(\frac{1}{1 - r_p}\right) = \prod_p \left(1 - \frac{f(p)}{p^s}\right)^{-1}$$

Ex $f(n) = n^a$ Cpl. Mult. $\Rightarrow$ $(a \in \mathbb{R})$

$$\sum_{n=1}^{\infty} \frac{1}{n^{s-a}} = \sum_{n=1}^{\infty} \frac{n^a}{n^s} = \prod_p \left(1 - \frac{p^a}{p^s}\right)^{-1} \prod_p \left(1 - \frac{1}{p^{s-a}}\right)^{-1}.$$

$\quad \|$

$\zeta(s-a)$

$Re(s) > a+1$

Dirichlet Product of Arithmetic Functions

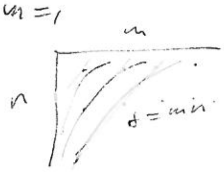
Def
$$(f * g)(n) := \sum_{d|n} f(d)\, g\left(\frac{n}{d}\right) = \sum_{d|n} f\left(\frac{n}{d}\right) g(d).$$

Prop 6.7 The Dirichlet Product of multiplicative functs is also mult.

Thm 6.9 Let
$$F(s) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s}, \qquad G(s) = \sum_{n=1}^{\infty} \frac{g(n)}{n^s}$$
have abscissa of abs conv α_a^f and α_a^g respectively.

Then if $s > \max(\alpha_a^f, \alpha_a^g)$ we get
$$F(s).G(s) = \sum_{n=1}^{\infty} \frac{(f*g)(n)}{n^s},$$
i.e. the product of functs rep. by Dirichlet Series is given by the D.S of their convolution of their coefficients.

Proof Since all series converge absolutely we can rearrange them at will. Then
$$F(s).G(s) = \left(\sum_{n=1}^{\infty} \frac{f(n)}{n^s}\right)\cdot\left(\sum_{m=1}^{\infty} \frac{g(m)}{m^s}\right) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{f(n)g(m)}{(nm)^s}$$



$$= \sum_{j=1}^{\infty} \sum_{nm=j} \frac{f(n)g(m)}{j^s}$$
$$= \sum_{j=1}^{\infty} \frac{1}{j^s}\left(\sum_{nm=j} f(n)g(m)\right) = \sum_{j=1}^{\infty} \frac{(f*g)(j)}{j^s}$$
$$= \sum_{n=1}^{\infty} \frac{(f*g)(n)}{n^s} \quad //$$

Ex. $\sum_{d|n} \varphi(d) = n \Rightarrow \zeta(s) \sum_{n=1}^{\infty} \frac{\varphi(n)}{n^s} = \left(\sum_{n=1}^{\infty} \frac{1}{n^s}\right)\cdot\left(\sum_{m=1}^{\infty} \frac{\varphi(m)}{m^s}\right)$

$(\Leftrightarrow (1 * \varphi)(n) = n)$ $\quad \sigma > 1 \quad \sigma > 2$

$$= \sum_{n=1}^{\infty} \frac{(1*\varphi)(n)}{n^s} = \sum_{n=1}^{\infty} \frac{n}{n^s}$$
$$= \sum_{n=1}^{\infty} \frac{1}{n^{s-1}} = \zeta(s-1).$$

Hence
$$\sum_{n=1}^{\infty} \frac{\varphi(n)}{n^s} = \frac{\zeta(s-1)}{\zeta(s)} \quad //$$
$\sigma > 2$

Ex 6.6 $\displaystyle\sum_{n=1}^{\infty} \frac{\varphi(n)}{n^s}$

$$= \prod_p \left(1 + \frac{(p-1)}{p} \left[\frac{p}{p^s} + \frac{p^2}{p^{2s}} + \frac{p^3}{p^{3s}} + \cdots \right] \right)$$

$$= \prod_p \left(1 + \left(\frac{p-1}{p}\right) \left[\frac{1}{p^{s-1}} + \frac{1}{p^{2(s-1)}} + \frac{1}{p^{3(s-1)}} \right] \right)$$

$$= \prod_p \left(1 + \frac{p-1}{p \cdot p^{s-1}} \left[\frac{1}{1 - \frac{1}{p^{s-1}}} \right] \right)$$

$$= \prod_p \left(\frac{1 - \cancel{\frac{1}{p^{s-1}}} + \cancel{\frac{1}{p^{s-1}}} - \frac{1}{p^s}}{1 - \frac{1}{p^{s-1}}} \right) = \prod_p \left(\frac{1 - \frac{1}{p^s}}{1 - \frac{1}{p^{s-1}}} \right) = \frac{\zeta(s-1)}{\zeta(s)} \quad //$$

Global Behaviour & Winters Thm. $\qquad f : \mathbb{N} \to \mathbb{C}$ arithmetic.

$$M(f) = \lim_{x \to \infty} \frac{\sum_{n \le x} f(n)}{x} \qquad \text{asymptotic mean value of } f(n).$$

If may or may not $\exists$.

Theorem 6.13 $\quad$ If. $\displaystyle\sum_{n=1}^{\infty} \frac{f(n)}{n^s} = \zeta(s) \sum_{n=1}^{\infty} \frac{g(n)}{n^s} \qquad (s > 1, \; s \in \mathbb{R})$.

If $\displaystyle\alpha = \sum_{n=1}^{\infty} \frac{g(n)}{n}$ converges absolutely $\boxtimes$ then $M(f)$ $\exists$ and $M(f) = \alpha$.

Proof $\quad$ Firstly $\displaystyle\zeta(s) \sum_{n=1}^{\infty} \frac{g(n)}{n^s} = \left(\sum_{n=1}^{\infty} \frac{1(n)}{n^s} \right) \cdot \left(\sum_{n=1}^{\infty} \frac{g(n)}{n^s} \right) \overset{\text{Thm 6.9}}{=} \sum_{n=1}^{\infty} \frac{(1 * g)(n)}{n^s}$

$\Longrightarrow f(n) = (1 * g)(n)$.

$$\therefore \sum_{n \le x} f(n) = \sum_{n \le x} \sum_{d \mid n} g(d) = \sum_{d \le x} g(d) \left\lfloor \frac{x}{d} \right\rfloor$$

$$= x \sum_{d \le x} \frac{g(d)}{d} - \sum_{d \le x} \left(\frac{x}{d} - \left\lfloor \frac{x}{d} \right\rfloor \right) g(d)$$

$$= x \sum_{d \le x} \frac{g(d)}{d} + O\Big(\underbrace{\sum_{d \le x} |g(d)|}_{} \Big)$$

$$=: x \underbrace{S_1(x)}_{} + O(S_2(x))$$

Now $\displaystyle S_1(x) = \sum_{d=1}^{\infty} \frac{g(d)}{d} + O\left(\sum_{d > x} \frac{g(d)}{d} \right) = \underbrace{\sum_{d=1}^{\infty} \frac{g(d)}{d} + o(1)}_{\alpha + o(1).}$ by $\boxtimes$.

And $\displaystyle S_2(x) = \sum_{d \le \sqrt{x}} \frac{|g(d)|}{d} \cdot d + \sum_{\sqrt{x} < d \le x} \frac{|g(d)|}{d} \cdot d$

$$\le \sqrt{x} \sum_{d \le \sqrt{x}} \frac{|g(d)|}{d} + x \sum_{\sqrt{x} < d \le x} \frac{|g(d)|}{d} = O(\sqrt{x}) + o(x) = o(x) \quad //$$

Average Order

Def^n If. $\exists$ an increasing function $\rho(n)$ such that
$$\frac{1}{x}\sum_{n\le x} f(n) \sim \frac{1}{x}\sum_{n\le x}\rho(n)$$ we say $\rho(n)$ is the average order of $f(n)$.

Ex $$\sum_{n\le x} d(n) = x\log x + (2\gamma-1)x + O(\sqrt{x})$$

Let. $\rho(n) = \log n$. so. $$\sum_{n\le x}\rho(n) = x\log x - x + O(\log x).$$

Thus $$\frac{\left(\sum_{n\le x} f(n)/x\right)}{\left(\sum_{n\le x}\rho(n)/x\right)} = \frac{\log x + (2\gamma-1) + O\left(\frac{1}{\sqrt{x}}\right)}{\log x - 1 + O\left(\frac{\log x}{x}\right)} \longrightarrow 1.$$

so. $\log n$ is the average order of $d(n)$.

Def^n If $n = p_1^{\alpha_1}\cdots p_m^{\alpha_m}$, $\omega(n) = m = \#$ distinct prime divisors.
$\Omega(n) = \alpha_1 + \dots + \alpha_m =$ total # of prime divisors.

Ex The average order of $\omega(n)$ is $\log\log(n)$.

Note $(a,b)=1 \Rightarrow \omega(ab) = \omega(a)+\omega(b)$, a so-called additive fun.

We need $$\sum_{n\le x}\omega(n) = \sum_{n\le x}\sum_{p\mid n} 1 = \sum_{p\le x}\left\lfloor\frac{x}{p}\right\rfloor$$ $-\left\{\frac{x}{p}\right\}$
$$= x\sum_{n\le x}\frac{1}{p} + \sum_{p\le x}\left(\left\lfloor\frac{x}{p}\right\rfloor - \frac{x}{p}\right)$$
$$= x\sum_{n\le x}\frac{1}{p} + O(\pi(x))$$ by Chebyshev. or PNT.
$$= x\left(\log\log x + M + O\left(\frac{1}{\log x}\right)\right) + O\left(\frac{x}{\log x}\right)$$
$$= x\log\log x + Mx + O\left(\frac{x}{\log x}\right)$$

need $\sum_{n=2}^{x}\log\log n$

Prop 1.2 [Euler Maclaurin]   Let $f'(t)$ be integr. on $[y,x]$.

$$\sum_{y<n\le x} f(n) = \int_y^x f(t)\,dt + \int_y^x \{t\} f'(t)\,dt + (\lfloor x\rfloor - x) f(x) - (\lfloor y\rfloor - y) f(y)$$

So if $f(n)=\log\log n$ and $y=2$.

$$\sum_{1<n\le x} \log\log n = \log\log 2 + \sum_{2<n\le x}\log\log n = \log\log 2 + \int_2^x \log\log t\,dt + \int_2^x \frac{\{t\}}{t\log t}\,dt$$

$(\Leftrightarrow 2\le n\le x)$

$$+ (\lfloor x\rfloor - x)\log\log x - 0 \qquad (y=2).$$

$\int \log\log t$

$$= x\log\log x - \int_2^x \frac{dt}{\log t} + O(\log\log x) + O(\log\log x)$$

slower than $\sqrt{x}$

$$= x\log\log x + O\left(\frac{x}{\log x}\right) + O(\log\log x).$$

So the average order of $\omega(n)$ is $\log\log(n)$, i.e. "most" numbers have very few distinct prime factors, i.e. "about" $\log\log n$.

Ex. $\Omega(n)$ is completely additive i.e. $\Omega(ab)=\Omega(a)+\Omega(b)\ \forall a,b\in\mathbb{N}$. What is its order?

& $\Omega(n)\geq\omega(n)$ with $\Omega(n)=\omega(n) \Leftrightarrow n$ is squarefree.

$$\sum_{n\le x}\Omega(n) = \sum_{n\le x}\sum_{\substack{p^k\mid n\\ k\ge1}} 1 = \sum_{\substack{p^k\le x\\ k\ge1}}\left\lfloor \frac{x}{p^k}\right\rfloor = \sum_{p\le x}\left\lfloor \frac{x}{p}\right\rfloor + \sum_{\substack{p^k\le x\\ k\ge2}}\left\lfloor \frac{x}{p^k}\right\rfloor$$

$$= x\sum_{p\le x}\frac{1}{p} - \sum_{p\le x}\left\{\frac{x}{p}\right\} + x\sum_{\substack{p^k\le x\\ k\ge2}}\frac{1}{p^k} - \sum_{\substack{p^k\le x\\ k\ge2}}\left\{\frac{x}{p^k}\right\}$$

$$= x\left(\log\log x + M + O\left(\frac{1}{\log x}\right)\right) + O(\pi(x)) + \underbrace{x\sum_p\sum_{k\ge2}\frac{1}{p^k}}_{x\sum_p \frac{1}{p(p-1)}} - x\underbrace{\sum_{k\ge2}\sum_{p^k>x}\frac{1}{p^k}}_{\otimes} + O\Big(\underbrace{\sum_{\substack{p^k\le x\\ k\ge2}}1}\Big)$$

$$\pi(x^{\frac{1}{2}}) + \pi(x^{\frac{1}{3}}) + \dots \ll \log x\,\pi(x^{\frac{1}{2}}) \ll \sqrt{x}$$

$$= x\left(\log\log x + \beta + \sum_p \frac{1}{p(p-1)}\right) + O\left(\frac{x}{\log x}\right)$$

So, again, the average order of $\Omega(n)$ is also $\log\log n$.

But $\Omega(3^m) = m$ and $\log\log 3^m = \log(m\log 3) = \log m + \log\log 3 = O(\log m)$ much less than m. How often is $\Omega(n)$ close to $\log\log n$?

Def^n Normal Order. Given $f(n)$ we say it has a normal order $\rho(n)$ if $\rho(n)$ is positive and increasing and $\forall \varepsilon > 0$:

$$\lim_{x\to\infty} \frac{1}{x} \# \left\{ n \le x : \left| \frac{f(n)}{\rho(n)} - 1 \right| < \varepsilon \right\} = 1.$$

Then $f(n) \sim \rho(n)$ "except for a set of integers of 0 density"

Then both $\omega(n)$ and $\Omega(n)$ have normal order $\log\log n$. (see below).

Thm 7.2. $$\sum_{n\le x} (\omega(n) - \log\log x)^2 = O(x \log\log x).$$

Proof (see back) $$\sum_{n\le x} \omega(n) = x \log\log x + O(x).$$

Claim $$\sum_{n\le x} \omega(n)^2 = x(\log\log x)^2 + O(x \log\log x). \quad (1)$$

now LHS $$= \sum_{n\le x} \left(\sum_{p|n} 1 \right)^2 = \sum_{n\le x} \left(\sum_{p|n} 1 + 2 \sum_{\substack{p<q \\ pq|n}} 1 \right) = S_1 + 2S_2$$

(easy) $$S_1 := \sum_{n\le x} \sum_{p|n} 1 = \sum_{n\le x} \omega(n) = x \log\log x + O(x) = O(x \log\log x) \quad (1')$$

(hard) $$S_2 := \sum_{\substack{p<q \\ pq\le x}} \sum_{\substack{n\le x \\ pq|n}} 1 = \sum_{\substack{p<q \\ pq\le x}} \left\lfloor \frac{x}{pq} \right\rfloor = \sum_{\substack{p<q \\ pq\le x}} \left(\frac{x}{pq} + O(1) \right) = x \sum_{\substack{p<q \\ pq\le x}} \frac{1}{pq} + O\left(\sum_{\substack{p<q \\ pq\le x}} 1 \right)$$

Let $N(x) := \#\{ n \le x : n = pq, p<q \}$ so the error is $O(N(x))$.

Now $p^2 < pq \le x \Rightarrow p \le \sqrt{x}$. Thus.

$$N(x) \le \sum_{p\le\sqrt{x}} \pi\left(\frac{x}{p}\right) \quad \text{since } q \le \frac{x}{p}.$$

$$\ll \sum_{p\le\sqrt{x}} \frac{x}{p \log(x/p)} \le \sum_{p\le\sqrt{x}} \frac{x}{p \log(\sqrt{x})} \quad \text{since } p \le \sqrt{x} \Rightarrow \sqrt{x} \le \frac{x}{p}.$$

$$= \frac{2x}{\log x} \sum_{p\le\sqrt{x}} \frac{1}{p} \ll \frac{x \log\log\sqrt{x}}{\log x} = o(x) \text{ as } x \to \infty.$$

Hence $$S_2 = x \sum_{\substack{p<q \\ pq\le x}} \frac{1}{pq} + o(x) \quad (2)$$

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Now $\displaystyle\sum_{\substack{p<q \\ pq \le x}} \frac{1}{pq} = \frac{1}{2}\left(\Big(\sum_{p\le x} \frac{1}{p}\Big)^2 - \sum_{p \le x} \frac{1}{p^2} - 2 \sum_{\substack{p<q\le x \\ pq > x}} \frac{1}{pq} \right)$

$$=: \frac{1}{2}\left(S_3^2 - S_4 - 2S_5 \right) \qquad (3')$$

We estimate S_3, S_4, S_5 separately:

$$S_3 = \sum_{p\le x} \frac{1}{p} = \log\log x + O(1) \qquad (\text{Thm } 4.5) \qquad (3)$$

$$\Rightarrow S_3^2 = (\log\log x)^2 + O(\log\log x)$$

And $\displaystyle S_4 = \sum_{p\le x} \frac{1}{p^2} < \sum_{p} \frac{1}{p^2} \le \sum_{n=1}^{\infty} \frac{1}{n^2} = O(1).$ (4)

Finally in S_5 we have $\sqrt{x} < q \le x$ so

$$S_5 \le \Big(\sum_{p\le x} \frac{1}{p}\Big) \cdot \Big(\sum_{\sqrt{x}<q\le x} \frac{1}{q} \Big) = \big(\log\log x + O(1)\big)\big(\log 2 + O(1)\big)$$

$$= O(\log\log x). \qquad (5)$$

So by (2) + (3) + (4) + (5)

$$S_2 = \frac{x}{2}\Big((\log\log x)^2 + O(1) + O(1) + O(\log\log x)\Big) + o(x) + \tfrac{1}{2}(\log\log x)^2$$

$$= \frac{x}{2}(\log\log x)^2 + O(x\log\log x). \qquad (6)$$

Hence $\displaystyle\sum_{n\le x} \omega(n)^2 \overset{(1'')}{=} S_1 + 2S_2$

$$\overset{(1')(6)}{=} O(x\log\log x) + x(\log\log x)^2 + O(x\log\log x)$$

$$= x(\log\log x)^2 + O(x\log\log x),\ \text{proving the } \underline{\text{claim}}.$$

Now (see back) $\displaystyle\sum_{n\le x} \omega(n) = x\log\log x + O(x)$

Hence $\displaystyle\sum_{n\le x} (\omega(n) - \log\log x)^2 = (\log\log x)^2 \lfloor x \rfloor + \sum_{n\le x} \omega(n)^2 - 2\log\log x \sum_{n\le x} \omega(n)$

$$= x(\log\log x)^2 + O((\log\log x)^2) + x(\log\log x)^2 + O(x\log\log x)$$
$$\qquad - 2x(\log\log x)^2 + O(x\log\log x)$$

$$= O(x\log\log x). \quad //$$

Thm 7.4 The function $\omega(n)$ has a <u>normal order</u> $\log\log n$.

<u>Proof</u>. By Thm 7.2 $\exists\, C>0$ so. for $x \geq x_0$.

$$\frac{1}{x}\sum_{n\leq x}(\omega(n)-\log\log x)^2 < C\log\log x.$$

Let $\sqrt{x}\leq n\leq x$ and suppose x is sufficiently large. Then.

(1) $\log\log x - 1 < \log\log x - \log 2 = \log\log\sqrt{x} \leq \log\log n \leq \log\log x \implies \underline{\log\log n = \log\log x + O(1)}$.

and (2) $n = p_1^{\alpha_1}\cdots p_\ell^{\alpha_\ell} \geq 2^\ell \implies \ell\log 2 \leq \log n \leq \log x \implies \ell = \omega(n) \ll \log x$.

Hence
$$\sum_{n\leq x}(\omega(n)-\log\log n)^2 = \sum_{\sqrt{x}\leq n\leq x}(\omega(n)-\log\log n)^2 + \sum_{n<\sqrt{x}}(\omega(n)-\log\log n)^2$$

$$= \sum_{\sqrt{x}\leq n\leq x}(\omega(n)-\log\log x + O(1))^2 + O(\sqrt{x}(\log x)^2)$$

$$= \sum_{\sqrt{x}\leq n\leq x}(\omega(n)-\log\log x)^2 + O\Big(\sum_{n\leq x}(\omega(n)+\log\log x)\Big) + O(x) + O(\sqrt{x}(\log x)^2)$$

Thm 7.2

$$= O(x\log\log x) + O(x\log\log x) + O(x\log\log x) + O(x) + O(\sqrt{x}(\log x)^2)$$

$$= O(x\log\log x)$$

(+)

$$\therefore\ \frac{1}{x}\sum_{n\leq x}(\omega(n)-\log\log n)^2 \leq C\log\log x \quad \text{also.}$$

It suffices to show that for sufficiently small $\delta>0$, as $x\to\infty$

$$h_\delta(x) = \#\{n\leq x : |\omega(n)-\log\log n| > (\log\log n)^{\frac{1}{2}+\delta}\} = o(x) \quad \boxed{K}$$

If <u>not</u> then. $h_\delta(x) \geq \alpha x$ for some $\alpha>0$ and <u>an ∞ number of $x\to\infty$</u>.

But in that case

$$C x\log\log x \overset{(+)}{\geq} \sum_{n\leq x}(\omega(n)-\log\log n)^2 \geq \sum_{\substack{\sqrt{x}\leq n\leq x\\ |\omega(n)-\log\log n| > \log\log n^{\frac{1}{2}+\delta}}}(\omega(n)-\log\log n)^2$$

$$\geq (\alpha x)(\log\log\sqrt{x})^{1+2\delta} \quad \text{a contradiction.} \quad //$$

Defn We say $f(n) = g(n)$ a.e. (almost everywhere) if $f(n) \neq g(n)$ at most on a set of density 0.

Thm 7.7 $\quad d(n) = \sum_{d|n} 1 = (\log n)^{\log 2 + o(1)}$ a.e.

Proof: Let $\delta > 0$ be given, $0 < \delta < \frac{1}{2}$, Let $A \subset \mathbb{N}$ be such that $n \in A \iff$.

$\boxtimes$ $\{n : |\Omega(n) - \log\log n| \leq (\log\log n)^{\frac{1}{2}+\delta}\}$ has density 1. Thus.

$$2^{\Omega(n)} = 2^{\log\log n + \Omega(n) - \log\log n} = 2^{\log\log n + O(\log\log n^{\frac{1}{2}+\delta})} \text{ . a.e}$$

$$= 2^{\log\log n \{1 + O(\frac{1}{\log\log n^{\frac{1}{2}-\delta}})\}} = 2^{\log\log n (1 + o(1))} \text{ a.e}$$

But $n = p_1^{\alpha_1} \dots p_m^{\alpha_m} \Rightarrow \quad d(n) = \prod_{i=1}^{m}(1+\alpha_i) \geq \prod_{i=1}^{m} 2 = 2^{\omega(n)}$.

So $\boxed{2^{\omega(n)} \leq d(n) \leq 2^{\Omega(n)}} \Leftarrow \quad d(n) = \prod_{i=1}^{m}(1+\alpha_i) \leq \prod_{i=1}^{m} 2^{\alpha_i} = 2^{\alpha_1 + \dots + \alpha_m} = 2^{\Omega(n)}$

Hence. $\quad d(n) \leq e^{\log 2 \log(\log n)(1 + o(1))} = (\log n)^{(1+o(1))\log 2}$ a.e.

Similarly, since $\boxtimes$ is also correct for $\omega(n)$ we get

$(\log n)^{(1+o(1))\log 2} = 2^{\omega(n)} \leq d(n)$ a.e. so therefore, since the union of two subsets of density zero has density zero, we get

$$d(n) = (\log n)^{(1+o(1))\log 2} \text{ a.e.} \quad //$$

But can we get complete results e.g. inequalities holding everywhere?

Ex $\quad n = p_1^{\alpha_1} \dots p_m^{\alpha_m} \geq 2^{\alpha_1 + \dots + \alpha_m} = 2^{\Omega(n)} \Rightarrow$

$$1 \leq \Omega(n) \leq \frac{\log n}{\log 2}$$

& $\Omega(p) = 1$ so 1 is best possible lower bound.

Prop 7.10 $\quad (\forall \varepsilon > 0)\,(\exists\, y_\varepsilon \text{ in } \mathbb{N})$ so $\forall\ n \geq y_\varepsilon$.

$$\omega(n) < \frac{1}{(1-\varepsilon)^2}\,\frac{\log n}{\log\log n} \quad \text{i.e.} \quad \omega(n) \ll_\varepsilon \frac{\log n}{\log\log n} \text{ as } n \to \infty.$$

Proof: $n = p_1^{\alpha_1} \cdots p_m^{\alpha_m}$. $\quad P_1 = 2,\ P_2 = 3,\ \dots$ so $p_1 \leq p_2 \dots \Rightarrow$

& $P_i \leq p_i \ \forall i = 1, 2, \dots$

$$\therefore \quad n \geq \prod_{i=1}^{m} P_i = \exp\Big(\sum_{i \leq m} \log P_i\Big)$$

$$\therefore \quad \log n \geq \sum_{i=1}^{m} \log P_i = \sum_{p \leq P_m} \log p = \theta(P_m) = P_m(1 + o(1)) \text{ by the PNT.}$$

$(\theta(x) \sim x)$.

But $P_m = (1 + o(1))\, m \log m$ by $\boxed{\leftarrow}$ (Prop 5.6) so $\therefore$

$$\log n \geq (1 + o(1))\, m \log m.$$

$\therefore \ \forall \varepsilon > 0 \ \exists\ m_\varepsilon$ so $\log n \geq (1-\varepsilon)\, m \log m$ ⊗ $\ \forall\ m \geq m_\varepsilon$.

Case 1 $\quad m > (\log n)^{1-\varepsilon} \Rightarrow \log n \geq (1-\varepsilon)^2 m \log\log n$

$$\Rightarrow \omega(n) = m \leq \frac{1}{(1-\varepsilon)^2}\,\frac{\log n}{\log\log n} \quad (1)$$

Case 2 $\quad m \leq (\log n)^{1-\varepsilon}$. $\quad$ for n suff large.

$$(\log n)^{1-\varepsilon} < \frac{1}{(1-\varepsilon)^2}\,\frac{\log n}{\log\log n}$$ so (1) is also true.

$$\text{so} \ \therefore \quad \omega(n) = m \leq \frac{1}{(1-\varepsilon)^2}\,\frac{\log n}{\log\log n}$$

$$\frac{\log\log n}{\log^{\varepsilon} n} \to 0$$

How about $d(n)$ (with no a.e.), $d(n) \geq 1$ upper bound? $d(n) \leq n^{1}$.

Prop 7.12

$$d(n) = n^{O\left(\frac{1}{\log\log n}\right)} \quad \forall\, n \geq 3 \quad \left\{ \begin{array}{l} \text{i.e. } d(n) \ll n^{\varepsilon} \text{ as } n \to \infty \\ \text{So } d(n) \text{ is "small"} \end{array} \right\}.$$

Proof: $n = \prod_{i=1}^{m} p_i^{\alpha_i} \Rightarrow$

$$\log d(n) = \sum_{i=1}^{m} \log(\alpha_i + 1) = \sum_{i=1}^{m} \left[\log((\alpha_i+1)\log(i+1)) - \log\log(i+1) \right]$$

$$= \sum_{i=1}^{m} \log((\alpha_i+1)\log(i+1)) - \sum_{i=1}^{m} \log\log(i+1) =: S_1 - S_2 \quad \text{—} (1)$$

First, estimate S_1: $\left(\prod_1^m x_i\right)^{\frac{1}{m}} \leq \frac{\sum_{i=1}^m x_i}{m}$ is AGM. Take logs —

$$\Rightarrow \frac{1}{m}\left(\sum_{i=1}^{m} \log x_i\right) \leq \log\left(\frac{1}{m}\sum_{i=1}^{m} x_i\right)$$

Then let $x_i = (\alpha_i + 1)(\log(i+1))$ to get

$$S_1 \leq m \log\left(\frac{1}{m}\sum_{i=1}^{m} (\alpha_i+1)\log(i+1)\right)$$

$$\leq m \log\left(\frac{1}{m}\sum_{i=1}^{m} \log p_i^{\alpha_i+1}\right)$$

order the primes $p_1 \leq p_2 \ldots$
$i+1 \leq P_i \leq p_i$
where $P_1 = 2$, $P_2 = 3$, ... are the seq. of primes

$$\leq m \cdot \log\left(\frac{\log(n^2)}{m}\right)$$

$$= m \log\left(\frac{2\log n}{m}\right)$$

$$= m \log\log n - m\log m + O(m) \quad \text{—} (2).$$

Now, estimate S_2: $\left\{ \begin{array}{l} x = m \\ a_n = 1 \\ f(t) = \log\log(t+1) \end{array} \right.$ Abel Summation $\Rightarrow$.

$$S_2 = \sum_{i=1}^{m} \log\log(i+1) = m\log\log(m+1) - \int_1^m \frac{[t]}{(t+1)\log(t+1)}\,dt.$$

$$= m\log\log(m+1) + O\left(\int_2^m \frac{dt}{\log t}\right)$$

$$= m\log\log(m+1) + O(m). \quad \text{—} (3).$$

From Prop 7.10 we can write $m := \omega(n) \ll \frac{\log n}{\log\log n}$

So by (1) + (2) + (3) we get

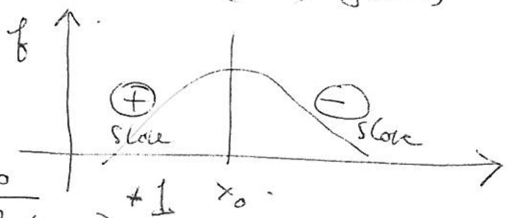
$$\log(d(n)) \le \underbrace{m\log\log n - m\log m - m\log\log(m+1)}_{f(m)} + O(m) = f(m) + O\left(\frac{\log n}{\log\log n}\right)$$

$$f(x) := x\log\log n - x\log x - x\log\log(x+1)$$

$$\Rightarrow f'(x) = \log\log n - 1 - \log x - \log\log(x+1) - \frac{x}{(x+1)\log(x+1)}$$. Let n be large.

$f'(x_0) = 0 \iff$.

$$\log\log n \overset{\boxtimes}{=} \log x_0 + \log\log(x_0+1) + \frac{x_0}{(x_0+1)\log(x_0+1)} + 1$$



So. $\forall m$ $$f(m) \le f(x_0) = x_0\left(\log\log n - \log x_0 - \log\log(x_0+1)\right)$$
$$\overset{\boxtimes}{=} x_0\left(1 + \frac{x_0}{(x_0+1)\log(x_0+1)}\right) = O(x_0)$$

By $\boxtimes$ $$\log\log n \ge \log x_0 + \log\log x_0 \Rightarrow \log n \ge x_0\log x_0 \Rightarrow x_0 \ll \frac{\log n}{\log\log n}$$
$$= \log(x_0\log x_0).$$

$\boxtimes$ $\log\log n \le 2\log x_0$ X

$$\therefore \log d(n) \le f(x_0) + O\left(\frac{\log n}{\log\log n}\right) + O(1) \ll x_0 \ll \frac{\log n}{\log\log n}$$

$$\therefore d(n) \ll e^{\log n \cdot \frac{1}{\log\log n}} = n^{\frac{1}{\log\log n}}$$

$$d(n) \le C \cdot n^{\frac{1}{\log\log n}} = n^{\frac{1}{\log\log n} + \frac{c'}{\log\log n}}$$
$$= n^{O\left(\frac{1}{\log\log n}\right)} //$$

$$\begin{cases} e = n^{\frac{1}{\log n}} \\ C = e^{\log C} = n^{\frac{\log C}{\log n}} \end{cases}$$

Example (Prop 7.16).

2

Let $A := \{n \in \mathbb{N} : p^2 \mid n$ for some prime $p > \log\log n\}$.

Then A has asymptotic density zero!.

Proof Let $x \in \mathbb{R}^+$ & $n \le x$ & x reasonably large.

Let $\frac{x}{\log x} \le n \le x$. $B = \{n \le \frac{x}{\log x}\}$ has density 0 so we study $A \setminus B$.

Then $\log\log n > \log\log\left(\frac{x}{\log x}\right) = \log(\log x - \log\log x)$

$= \log\left(\log x\left[1 - \frac{\log\log x}{\log x}\right]\right)$.

$= \log\log x + \log\left(1 - \frac{\log\log x}{\log x}\right) > \frac{1}{2}\log\log x$

as $x \to \infty$.
i.e. $x \geq x_0$

Let $n \in A \setminus B \Rightarrow p^2 \mid n$ & $p > \log\log n > \frac{\log\log x}{2}$

Fix such a p. The number of $n \in A \setminus B$ with $p^2 \mid n$ is at most $\left\lfloor \frac{x}{p^2} \right\rfloor$.

Thus
$$\#(A \setminus B) \le \sum_{p > \frac{\log\log x}{2}} \frac{x}{p^2} \le x \int_{\frac{\log\log x}{2} - 1}^{\infty} \frac{dt}{t^2}$$

$$= \frac{x}{\frac{\log\log x}{2} - 1} = o(x)$$

//.

Example (Prop 7.19) How many numbers occur in the multiplication table of size $N \times N$?

	1	2	3	4
1	1	2	3	4
2	2	4	6	8
3	3	6	9	12
4	4	8	12	16

$A(4) = 9$

$A(100) = 2906$

$A(N) = \#$ distinct $\oplus$ integers in the $N \times N$ table

set $\{k\ell : 1 \le k \le N,\ 1 \le \ell \le N\}$

Claim: Then $\frac{A(N)}{N^2} \to 0$ as $N \to \infty$ i.e. $A(N) = o(N^2)$.

Given $\varepsilon > 0$, other than a set of size $o(N)$, all $k \le N$ have

$\Omega(k) \in [(1-\varepsilon)\log\log N, (1+\varepsilon)\log\log N]$ & same for $\Omega(\ell)$ ✓

Thus $\Omega(k\ell) = \Omega(k) + \Omega(\ell) \in [(2-2\varepsilon)\log\log N, (2+2\varepsilon)\log\log N]$

But "most" integers n in $[1, N^2]$ have $\Omega(n) \in [(1-\varepsilon)\log\log(N^2), (1+\varepsilon)\log\log N^2]$ &

$\log\log(N^2) = \log(2\log N) = \log\log N + \log 2 = \log\log N(1 + o(1))$.

$(1-\varepsilon)\log\log N^2$ $\quad$ $(2-2\varepsilon)\log\log N$

If N is sufficiently large & then intervals are disjoint.
& $\varepsilon < \frac{1}{3}$

$1 + \varepsilon < 2 - 2\varepsilon \iff$
$3\varepsilon < 1 \iff \varepsilon < \frac{1}{3}$.

//

Ch 8 The Euler Phi Function

$$\phi(n) = |\{ j : 1 \le j \le n \;\&\; (j,n) = 1\}|.$$

Then the set of classes $\mathbb{Z}/n\mathbb{Z}$ is a ring and its units $(\mathbb{Z}/n\mathbb{Z})^*$ is an abelian group of order $\varphi(n)$. Hence.

If $(a,n) = 1$ so $[a] \in (\mathbb{Z}/n\mathbb{Z})^*$.

$$\boxed{a^{\phi(n)} \equiv 1 \bmod n.} \quad \text{(Euler's Theorem).}$$

Let $(m_i, m_j) = 1$ $i \ne j$ $1 \le i, j \le k$. & let $M = m_1 \cdots m_k$.

Define a map

$$\varphi : \mathbb{Z}/(m_1 \cdots m_k \mathbb{Z}) \longrightarrow (\mathbb{Z}/m_1\mathbb{Z}) \times \cdots \times (\mathbb{Z}/m_k\mathbb{Z}) \quad \boxtimes$$

$$[a]_{m_1 \cdots m_k} \longrightarrow ([a]_{m_1}, \ldots, [a]_{m_k}).$$

Then φ is a ring isomorphism.

① φ is a homomorphism of rings

$$\begin{aligned}
\varphi([a]_M + [b]_M) &= \varphi([a+b]_M) && (\text{defn } +_M.)\\
&= ([a+b]_{m_1}, \ldots) && (\text{defn } \varphi)\\
&= ([a]_{m_1} + [b]_{m_1}, \ldots) && (\text{defn } +_{m_i}.)\\
&= ([a]_{m_1}, \ldots) + ([b]_{m_1}, \ldots) && (\text{defn vector } +)\\
&= \varphi([a]_M) + \varphi([b]_M) && (\text{defn } \varphi).
\end{aligned}$$

Similarly $\varphi([ab]_M) = \varphi([a]_M)\,\varphi([b]_M)$.

② φ is injective: Let $\varphi([a]_M) = 0 = (0, \ldots 0) \Rightarrow [a]_{m_i} = 0 \;\forall i \; 1 \le i \le k$

$\Rightarrow m_i | a$ for $1 \le i \le a \Rightarrow M | a \Rightarrow [a]_M = 0$.

③ Let $([r_1]_{m_1}, \ldots [r_k]_{m_k}) \in (\mathbb{Z}/m_1\mathbb{Z}) \times \cdots$

By the CRA $\exists\, a \in \mathbb{Z}$ so $a \equiv r_i \bmod m_i$. Then

$$\varphi([a]_M) = ([a]_{m_i}, \ldots) = ([r_i]_{m_i}, \ldots) \quad \text{so } \varphi \text{ is surjective} //$$

Hence φ is an isomorphism & so the multiplic. groups of the two rings in $\boxtimes$ are isomorphic $\Rightarrow$ their sizes are equal so.

$$\varphi(m_1 \cdots m_k) = \varphi(m_1) \cdots \varphi(m_k) \Rightarrow \varphi \text{ is multiplicative} //$$

Prop 8.2 $\quad n = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$ standard prime decomp. Then

$\sigma(n) = \sum_{d|n} d$ sum of divisors

$$\varphi(n) = n \prod_{i=1}^{k} \left(1 - \frac{1}{p_i}\right), \quad \text{and.}$$

$$\sigma(n) = \prod_{i=1}^{k} \left(\frac{p_i^{\alpha_i+1} - 1}{p_i - 1} \right)$$

Proof: Let $n = p^\alpha$. Then if $(n,a) \neq 1$ & $a \le n$ we have $p|a \Rightarrow$ $a \in A = \{p, 2p, 3p, \ldots (p^{\alpha-1})p\}$ the set of all multiples of $p \le p^\alpha$ & $|A| = p^{\alpha-1}$.

Hence $\varphi(p^\alpha) = p^\alpha - p^{\alpha-1} = p^\alpha(1 - \frac{1}{p})$. Hence if $n = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$

$$\varphi(n) = \prod_{p|n} p^\alpha\left(1 - \frac{1}{p}\right) = n \prod_{p|n}\left(1 - \frac{1}{p}\right).$$

Again if $n = p^\alpha$, $\sigma(n) =$ sum of divisors of n
$= 1 + p + p^2 + \cdots p^\alpha = \frac{p^{\alpha+1} - 1}{p-1}$, sum of a geom series.

and the result follows since σ is multiplicative $\boxed{\sigma = \tau * id}$, $\tau(n) := 1$

Ex $\quad \forall n \, \exists m$ so $\varphi(m) = n!$ (Erdős).

Proof: Since $\prod_{p \le n} (p-1)$ is the product of distinct integers all $\le n$ we have

$$\prod_{p\le n}(p-1) \mid n!$$

Also $\quad n! = \prod_{p \le n} p^{\alpha_p} \qquad \alpha_p \geq 1.$

$$\Rightarrow \quad \frac{n!}{\prod_{p\le n}(p-1)} = \prod_{p\le n} p^{\beta_p} \qquad \beta_p \geq 0.$$

Let $\quad m := \prod_{p \le n} p^{\beta_p + 1}.$

$$\Rightarrow \quad \varphi(m) = \prod_{p\le n}\left(p^{\beta_p}(p-1)\right) = n! \quad //$$

now $1 \le \varphi(n) \le n-1$, $\forall\, n \geqslant 2$.

Prop 8.4

$\forall\, n \geqslant 3$. $\quad \varphi(n) \gg \dfrac{n}{\log\log n}$.

Proof. Let $n = q_1^{\alpha_1} \cdots q_k^{\alpha_k}$. $\quad q_1 < q_2 \cdots < q_k$ & $\alpha_j > 0$.

Let p_i be the ith prime $p_1 = 2$, $p_2 = 3$, ... Then. $q_i \geqslant p_i \;\forall i \Rightarrow$

$$\varphi(n) = n \prod_{i=1}^{k}\left(1 - \frac{1}{q_i}\right) \geqslant n \prod_{i=1}^{k}\left(1 - \frac{1}{p_i}\right) = n \prod_{p \le p_k}\left(1 - \frac{1}{p}\right) \gg \frac{n}{\log p_k} \text{ by}$$

Mertens Estimate.

But. $k = \omega(n) \ll \log n$ (Prop 7.10). & $p_k \sim k \log k \Rightarrow p_k \ll k^2$

$$\Rightarrow \quad \varphi(n) \gg \frac{n}{\log p_k} \geqslant \frac{n}{\log(k^2)} \gg \frac{n}{\log k} \gg \frac{n}{\log\log n} \quad //$$

Prop 8.5 $\quad \sigma(n) \ll n \log\log n \; (\forall\, n \geqslant 3)$.

Proof: $\quad \varphi(n)\sigma(n) = \prod_i q_i^{\alpha_i} \prod_i \left(1 - \frac{1}{q_i}\right) \prod_i \dfrac{q_i^{\alpha_i+1} - 1}{q_i - 1}$.

Let $n = q_1^{\alpha_1} \cdots q_k^{\alpha_k}$.

$$= \prod_i q_i^{\alpha_i - 1} \cancel{(q_i - 1)} \frac{(q_i^{\alpha_i+1} - 1)}{\cancel{(q_i - 1)}}$$

$$= \prod_i \left(q_i^{2\alpha_i} - q_i^{\alpha_i - 1}\right) \le \prod_i q_i^{2\alpha_i} = n^2$$

$$\therefore \quad \sigma(n) \le \frac{n^2}{\varphi(n)} \ll \frac{n^2}{n/\log\log n} = n \log\log n. \quad //$$

c/f.

$$\left(\sum_{n \le x} n = \frac{x^2}{2} + O(x)\right).$$

Thm 8.6 Average order of the Euler function

$$\sum_{n \le x} \varphi(n) = \frac{3}{\pi^2} x^2 + O(x \log x).$$

$$f(n) = \sum_{d|n} g(d) \Rightarrow g(n) = \sum_{d|n} \mu(d) f\left(\frac{n}{d}\right)$$

Proof. Möbius Inversion applied to.

$$n = \sum_{d|n} \varphi(d) \qquad \varphi(n) = \sum_{d|n} \mu(d) \left(\frac{n}{d}\right)$$

$$\Rightarrow \quad \boxed{\frac{\varphi(n)}{n} = \sum_{d|n} \frac{\mu(d)}{d}} \quad \text{— } 1$$

$$\therefore \quad B(x) := \sum_{n \le x} \frac{\varphi(n)}{n} = \sum_{n \le x} \sum_{d \mid n} \frac{\mu(d)}{d}$$

$$= \sum_{d \le x} \frac{\mu(d)}{d} \left\lfloor \frac{x}{d} \right\rfloor$$

$$= \sum_{d \le x} \frac{\mu(d)}{d} \left(\frac{x}{d} + O(1) \right)$$

$$= x \sum_{d \le x} \frac{\mu(d)}{d^2} + O\left(\sum_{d \le x} \frac{1}{d} \right)$$

$$= x \left(\sum_{d=1}^{\infty} \frac{\mu(d)}{d^2} - \sum_{d > x} \frac{\mu(d)}{d^2} \right) + O\left(\sum_{d \le x} \frac{1}{d} \right)$$

$$= x \sum_{d=1}^{\infty} \frac{\mu(d)}{d^2} + O\left(x \sum_{d > x} \frac{1}{d^2} \right) + O\left(\sum_{d \le x} \frac{1}{d} \right)$$

$$= x \sum_{d=1}^{\infty} \frac{\mu(d)}{d^2} + O(1) + O(\log x)$$

But: $\dfrac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \dfrac{\mu(n)}{n^s}$ & $\zeta(2) = \dfrac{\pi^2}{6} \implies$

$$B(x) = \frac{6}{\pi^2} x + O(\log x).$$

Now use Abel summation with $a_n = \dfrac{\varphi(n)}{n}$ and $f(t) = t$ so. $A(x) = B(x)$
$\Rightarrow f'(t) = 1$ &

$$\& \quad \sum_{n \le x} a_n f(n) = \sum_{n \le x} \varphi(n) = A(x) f(x) - \int_1^x A(t) f'(t)\, dt.$$

$$= x \left(\frac{6}{\pi^2} x + O(\log x) \right) - \int_1^x \left(\frac{6}{\pi^2} t + O(\log t) \right) dt.$$

$$= \frac{6}{\pi^2} x^2 + O(x \log x) - \frac{6}{2\pi^2} x^2 + O(1) + O(x \log x)$$

$$= \frac{3}{\pi^2} x^2 + O(x \log x).$$

7 8.4 $\varphi(n)\sigma(n) = \square$

Prop 8.7 $\quad |\{n \le x : \varphi(n)\sigma(n) = \square\}| \ge x^{\frac{c_1}{\log\log x}}$, $c_1 > 0$ $x_1 \to \infty$.

consequently there $\exists$ an ∞ number of positive integers n for which $\varphi(n)\sigma(n) = \square$.

Let. $A = \{p : p \le y\}$ be sets of primes.

$B = \{p : p \le \frac{y+1}{2}\}$

If $P \subset A$ define $n_P = \prod_{p\in P} p$ & note that

$$\varphi(n_P)\sigma(n_P) = \prod_{p\in P}(p-1)(p+1) =: m_P t_P^2 \quad \text{with } m_P \text{ squarefree}$$

$\therefore P \ni q \mid m_P \Rightarrow q \in B$.

Now A has $2^{\pi(y)}$ subsets, and B has $2^{\pi(\frac{y+1}{2})}$ subsets.

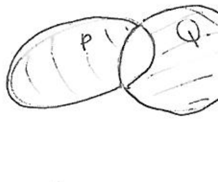
$\therefore$ there exists a family $\mathcal{F}$ of subsets P in A and a number m such that

$$\varphi(n_P)\sigma(n_P) = m t_P^2 \quad \text{———} \quad \boxtimes$$

& $$|\mathcal{F}| \ge \frac{2^{\pi(y)}}{2^{\pi(\frac{y+1}{2})}} = 2^{\frac{y}{2\log y}(1+o(1))},$$ by the PNT as $y \to \infty$.

Fix an element $Q \in \mathcal{F}$ and for $P \in \mathcal{F}$ def

$$P \Delta Q := (P\setminus Q) \cup (Q \setminus P)$$



"symmetric difference"

then $$\varphi(n_P)\sigma(n_P)\varphi(n_Q)\sigma(n_Q) = \varphi(n_{P\Delta Q})\sigma(n_{P\Delta Q})\left(\varphi(n_{P\cap Q})\sigma(n_{P\cap Q})\right)^2$$

and, by $\boxtimes$ $$\varphi(n_P)\sigma(n_P)\varphi(n_Q)\sigma(n_Q) = (m t_P t_Q)^2 = \square$$

$\therefore$ $$\varphi(n_{P\Delta Q})\sigma(n_{P\Delta Q}) = \square$$

$A \Delta B = A' \Delta B$ $\Rightarrow A \cap B^c = A' \cap B^c$. & $x \in A \cap B \Rightarrow x \notin A \Delta B \Rightarrow$ $x \notin A'\Delta B \Rightarrow x \in A' \cap B$.

Now the function $\theta(P) = P \Delta Q$. is injective $\mathcal{P}(A) \longrightarrow \mathcal{P}(A)$

$P \longmapsto n_{P\Delta Q}$ is also injective.

Hence $|\mathcal{F}| \ge 2^{\frac{y}{2\log y}(1+o(1))}$

$\therefore$ there are at least ~~elements~~ subsets P of A such that $\varphi(n_P)\sigma(n_P) = \square$

Let $x = e^{y(1+o(1))}$

and recall $\sum_{p \le y} \log p = y(1+o(1)) \Rightarrow \prod_{p \le y} p = e^{y(1+o(1))} = x$

$\therefore$ $y = (1+o(1)) \log x$ as $x \to \infty$, so $\therefore$

$$|\mathcal{F}| \geq 2^{\frac{y}{2\log y}(1+o(1))} = 2^{\frac{\log x}{2\log\log x}(1+o(1))} = x^{\frac{\log 2}{2\log\log x}(1+o(1))} \quad //$$

<u>Prop 8.8</u>. $D = \{ \frac{\varphi(n)}{n} : n \in \mathbb{N} \}$ is dense in $[0,1]$

<u>Proof</u> Let $P \subset \mathbb{P}$ be a finite set of primes,

Now $0 \leq \frac{\varphi(n)}{n} \leq 1$ so $\frac{n}{\varphi(n)} \geq 1$ and

& $\log \frac{n}{\varphi(n)} \geq 0$

so $\frac{\varphi(n)}{n}$ is dense in $[0,1] \iff \{ \log(\frac{n}{\varphi(n)}) : n \in \mathbb{N} \}$ is dense in $(0,\infty)$.

<u>But</u> $\log\left(\frac{n}{\varphi(n)}\right) = \log \prod_{p|n} \left(1 - \frac{1}{p}\right)^{-1} = \log \prod_{p|n}\left(1 + \frac{1}{p-1}\right) = \sum_{p|n} \log\left(1 + \frac{1}{p-1}\right)$

<u>Lemma</u> If $a_n > 0$ & $a_n \to 0$ and $\sum_{n=1}^{\infty} a_n = \infty$, then for every $x \in (0,\infty)$ & $\varepsilon > 0$ $\exists$ positive integers $n_1 < n_2 < \dots < n_s$ with $n_1 \geq N$.

$$|x - \sum_{i=1}^{s} a_{n_i}| < \varepsilon$$

(can N, ε & x)

<u>Proof</u> $\boxed{\sum_{N=n}^{\infty} a_n = \infty}$ & $a_n < \varepsilon \ \forall n \geq N+1$. Choose first $a_{N+1} + \dots + a_s > x$ then this works & $a_{N+1} + \dots + a_{s-1} \leq x$

$a_n = \log(1 + \frac{1}{p_n - 1})$ with p_n the n^{th} prime.

So given x and $\varepsilon > 0$, let

Finally. $0 < \log(1 + \frac{1}{p_n - 1}) < \frac{1}{p_n - 1} \Rightarrow a_n \to 0+$.

& $\sum_p \log(1 + \frac{1}{p-1}) \geq \sum_p \frac{1}{2(p-1)} \geq \sum_p \frac{1}{4p} = \infty$.

and the result follows //