Partitions: Bell and Stirling numbers

Ex $\{1,2,3,4,5\} = \{1,2,3\} \sqcup \{4\} \sqcup \{5\} =: [5]$

We say the RHS is a partition of the LHS; $[n] := \{1,\dots,n\}$.

Ex. All partitions of $\{1,2,3,4\}$ into two classes:

$\{1,2\} \sqcup \{3,4\}$, $\{1,3\} \sqcup \{2,4\}$, $\{1,2,3\} \sqcup \{4\}$, $\{1,2,4\} \sqcup \{3\}$
$\{1,3,4\} \sqcup \{2\}$, $\{1\} \sqcup \{2,3,4\}$, $\{1,4\} \sqcup \{2,3\}$ $\rightarrow$ seven partitions.

Let the number of partitions of $[n]$ into k classes be denoted

$\left\{ {n \atop k} \right\}$ with $\left\{ {n \atop k} \right\} = 0$ if $k > n$. Called

"Stirling numbers of the second kind". Set $\left\{ {n \atop 0} \right\} = 0$

Derivation of a recurrence relation

Consider all possible partitions of $[n]$ into k classes. Type (1) has $\{n\}$ as one of the partitions, type (2) has n included with other elements. For type (1) there are $\left\{ {n-1 \atop k-1} \right\}$ partitions.

For type (2) if we go through the subsets and erase the n wherever it appears we won't effect the number of classes, i.e. have partitions of $n-1$ symbols into k classes. But the same partition can appear several times e.g.

$\{1,2\} \sqcup \{3,4\}$, $\{1,3\} \sqcup \{2,4\}$, $\{1,4\} \sqcup \{2,3\}$, $\{1,2,4\} \sqcup \{3\}$, $\{1,3,4\} \sqcup \{2\}$, $\{1\} \sqcup \{2,3,4\}$

i.e. each partition occurs twice, or $2\left\{ {3 \atop 2} \right\}$ partitions

In general we have, for (2), partitions of $[n-1]$ into k classes with each partition written down k times (since the n could have been in any one of the k partition elements. i.e. $k\left\{ {n-1 \atop k} \right\}$

Hence
$$\boxed{\left\{ {n \atop k} \right\} = \left\{ {n-1 \atop k-1} \right\} + k\left\{ {n-1 \atop k} \right\}}$$

Also set $\genfrac{\{}{\}}{0pt}{}{n}{k} = 0$ if $k > n$ or $n < 0$ or $k < 0$

& $\genfrac{\{}{\}}{0pt}{}{n}{0} = 0, \; n \neq 0, \quad \genfrac{\{}{\}}{0pt}{}{0}{0} = 1$

then $\square$ is true $\forall \, (n,k) \neq (0,0)$.

Generating Functions

(A) $$B_k(x) = \sum_{n \geq 0} \genfrac{\{}{\}}{0pt}{}{n}{k} x^n$$

$\times \, \square$ by x^n and sum over n $\Rightarrow$

$$B_k(x) = x B_{k-1}(x) + k x B_k(x). \quad , \; k \geq 1, \; B_0(x) = 1.$$

$$\Rightarrow \quad B_k(x) = \left(\frac{x}{1-kx} \right) B_{k-1}(x).$$

$$\Rightarrow \quad \boxed{B_k(x) = \frac{x^k}{(1-x)(1-2x)\ldots(1-kx)} \qquad k \geq 0}$$

Partial fractions $$\frac{1}{(1-x)\ldots(1-kx)} = \sum_{j=1}^{k} \frac{\alpha_j}{(1-jx)}$$

fix r, $1 \leq r \leq k$, $\times$ both sides by $1-rx$ & let $x = \frac{1}{r}$ $\Rightarrow$

$$\alpha_r = \circ\circ\circ = (-1)^{k-r} \frac{r^{k-1}}{(r-1)!(k-r)!}, \quad 1 \leq r \leq k.$$

$$\Rightarrow \quad \genfrac{\{}{\}}{0pt}{}{n}{k} = \text{coeff of } x^n \text{ in } \frac{x^k}{(1-x)\ldots(1-kx)}$$

$$= \text{coeff of } x^{n-k} \text{ in } 1/[(1-x)\ldots(1-kx)]$$

$$= \text{coeff of } x^{n-k} \text{ in } \sum_{r=1}^{k} \frac{\alpha_r}{1-rx}$$

$$= \sum_{r=1}^{k} \alpha_r r^{n-k}$$

$$= \sum_{r=1}^{k} (-1)^{k-r} \frac{r^{k-1}}{(r-1)!\,(k-r)!} r^{n-k} = \sum_{r=1}^{k} (-1)^{k-r} \frac{r^n}{r!(k-r)!}.$$

$(n, k \geq 0)$

Check $\genfrac{\{}{\}}{0pt}{}{4}{2} = 7$.

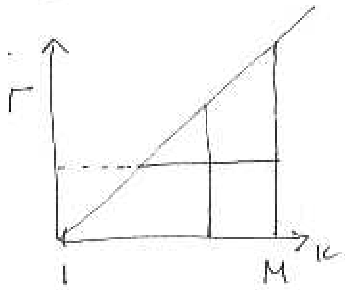
Bell numbers: the number of ways, $b(n)$, to partition a set of n elements. Let $b(0)=1$, $b(1)=1$

$$b(2) = 2 \leftarrow \{\{1\}\cup\{2\},\ \{1,2\}\}$$

Let M be any number with $M \geq n$ and then

$$b(n) = \sum_{k=1}^{M} \left\{ {n \atop k} \right\} \qquad \text{since } \left\{ {n \atop k} \right\} = 0 \text{ for } k>n,$$

$$= \sum_{k=1}^{M} \sum_{r=1}^{k} (-1)^{k-r} \frac{r^{n-1}}{(r-1)!(k-r)!}$$



$$= \sum_{r=1}^{M} \frac{r^{n-1}}{(r-1)!} \sum_{k=r}^{M} \frac{(-1)^{k-r}}{(k-r)!}$$

$$= \sum_{r=1}^{M} \frac{r^{n-1}}{(n-1)!} \left\{ \sum_{s=0}^{M-r} \frac{(-1)^s}{s!} \right\}$$

Keep n & r fixed and let $M \to \infty$ so $\{\ \} \to \frac{1}{e} \Rightarrow$

$$\boxed{b(n) = \frac{1}{e} \sum_{r\geq 0} \frac{r^n}{r!}} \qquad n \geq 0$$

Generating function

Let $$B(x) = \sum_{n \geq 0} \frac{b(n)}{n!} x^n$$ (an "exponential generating fun")

Hence by □ $$B(x) - 1 = \frac{1}{e} \sum_{n\geq 1} \frac{x^n}{n!} \sum_{r \geq 1} \frac{r^{n-1}}{(r-1)!}$$

$$= \frac{1}{e} \sum_{r\geq 1} \frac{1}{r!} \sum_{n\geq 1} \frac{(rx)^n}{n!}$$

$$= \frac{1}{e} \sum_{r \geq 1} \frac{1}{r!} (e^{rx} - 1)$$

$$= \frac{1}{e} (e^{e^x} - e) = e^{e^x - 1} - 1$$

$$\Rightarrow \quad B(x) = e^{e^x - 1}$$

$\Rightarrow$ $b(n)$ is the coeff of $\frac{x^n}{n!}$ in the expansion of $B(x) =$

$1 + x + x^2 + \frac{5x^3}{6} + \frac{5x^4}{8} + \frac{13x^5}{30} + O(x^6)$ e.g. $b(3) = 5$.

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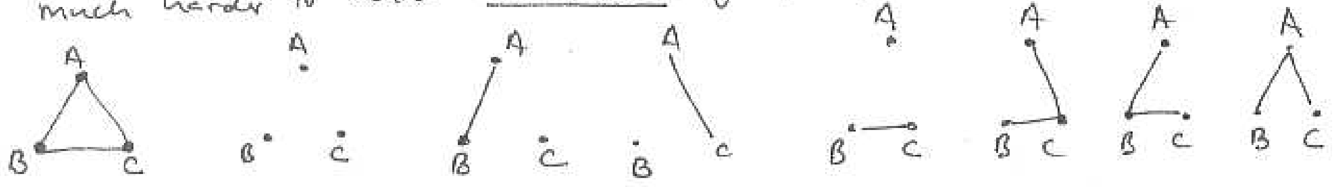
Pólya Redfield Enumeration

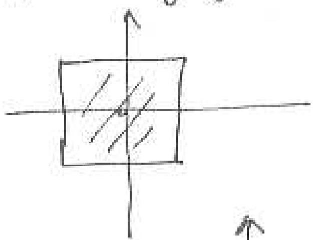
<u>Ex</u> <u>Labelled simple graphs</u> on n vertices is $2^{\frac{n(n-1)}{2}}$, but its much harder to count <u>unlabelled</u> graphs:



View unlabelled objects as <u>equivalence classes</u> of labelled objects.

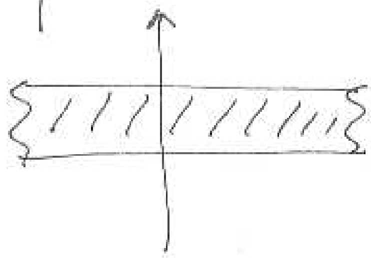
<u>Defn</u> A symmetry σ of $S \subset \mathbb{R}^2$ is a <u>rigid motion</u> (translation + rotation) such that after applying σ to S it <u>looks the same</u> as before.

e.g.



σ = rotate by $\frac{\pi}{2}$

e.g.



σ = translate by $(1,0)$

<u>or</u> $\sigma : S \to S$ is a symmetry if the distance between any pair of pts $x, y \in S$ is equal to the distance between $\sigma(x)$ and $\sigma(y)$, with σ a 1-1 and onto function

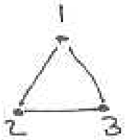
These symmetries form a <u>group</u> G

<u>i.e.</u> ① If σ and $\alpha \in G$ so does $\sigma \circ \alpha$

② If $\sigma \in G$ so does σ^{-1}

③ and the <u>identity</u> $e(x) = x \ \forall x \in S$ is a symmetry.

<u>Ex</u> S =



$G = \{$all permutations of $(1,2,3)\} = S_3$

Let a set of labelled objects be A and the symmetry group be G. Then two objects $a, b \in A$ are regarded as <u>equivalent</u> if $g(a) = b$ for some $g \in G$. This is an equivalence relation (reflexive, symmetric, transitive) and the equivalence classes are the subsets $\boxed{\text{Orbit}(a) := \{ g(a) : g \in G \} \subset A}$ for each $a \in A$.

<u>Ex</u>

$$\text{Orbit}\left(\overset{A}{\underset{BC}{\Delta}}\right) = \{ \Delta \} \qquad \text{when the labels are implicit.}$$

$$\text{orbit}\,(\therefore) = \{ \therefore \}$$

$$\text{orbit}\,(/.) = \{ /. \,,\, .\backslash \,,\, \therefore \}$$

$$\text{Orbit}\,(\lrcorner) = \{ \lrcorner \,,\, \llcorner \,,\, \wedge \}$$

To count the # of unlabelled objects we need to count the number of different <u>orbits</u>.

<u>Ex</u> At a picnic there are many families and we want to count the number of families. Ask <u>each</u> person "How <u>big</u> is your family?" and if the answer is k assign this person the wt $1/k$. Then

$$\#\,\text{families} = \sum_{\text{all people}} \frac{1}{k} \qquad \text{since a family of size } k \text{ gets a total wt of } 1.$$

Hence, by this reasoning.

$$\#\,\text{orbits} = \sum_{a \in A} \frac{1}{|\text{Orbit}(a)|} \quad \text{———} \quad (1)$$

<u>Defn</u> $\text{Fix}(a) := \{ g \in G : g(a) = a \}$ the stabilizer of $a \in A$.

If $b = g(a) \in \text{Orbit}(a)$ associate $g\,\text{Fix}(a) \subset G$, a 'coset'.

<u>Defn</u> $\left.\begin{aligned} \theta : G &\longrightarrow \text{Orbit}(a) \\ g &\longmapsto g(a) \end{aligned}\right\}$ This is <u>onto</u> and the number of elements in $\theta^{-1}(b)$ is $|\text{Fix}(a)|$ $\forall\, b \in \text{Orbit}(a)$

since $\theta^{-1}(g(a)) = g\,\text{Fix}(a)$ & $|g\,\text{Fix}(a)| = |\text{Fix}(a)|$.

$$\therefore \quad |\text{orbit}(a)| = \frac{|G|}{|\text{Fix}(a)|}$$

Hence, by (1)

$$\# \text{ orbits} = \sum_{a \in A} \frac{|\mathrm{Fix}(a)|}{|G|}$$

$$= \frac{1}{|G|} \sum_{a \in A} |\mathrm{Fix}(a)|$$

$$= \frac{1}{|G|} \sum_{a \in A} \sum_{g \in G} \chi(g(a)=a) \quad \text{where } \chi(?) = \begin{cases} 1 & ? \text{ true} \\ 0 & ? \text{ false} \end{cases}$$

$$= \frac{1}{|G|} \sum_{g \in G} \sum_{a \in A} \chi(g(a)=a)$$

$$= \frac{1}{|G|} \sum_{g \in G} \mathrm{fix}(g)$$

where $\mathrm{fix}(g) = \#\{a \in A : g(a) = a\}$ = # of elements of A fixed by $g \in G$.

<u>Theorem</u> [Burnside's Lemma, Cauchy, Frobenius] If G acts on A then

$$\boxed{\#\text{orbits} = \Big(\sum_{g \in G} \mathrm{fix}(g)\Big) \Big/ |G|}.$$

i.e. the number of orbits is the average number of fixed points of g for $g \in G$.

<u>Ex</u> $G = S_n = \{$all permutations of $(1, 2, \ldots, n)\}$, $1 = \sum_{\sigma \in S_n} \frac{\mathrm{fix}(\sigma)}{n!}$

with say $\sigma = (1)(2)(3)(4\,5\,6)$ fixing $\{1,2,3\} \subset \{1, \ldots, 6\}$

so $\mathrm{fix}(\sigma) = 3$

and $\tau = (1\,2\,3)(4\,5\,6) \Rightarrow \mathrm{fix}(\tau) = 0$.

<u>Ex</u> [Polya colouring] Let U be a set and C a set of colours.
A colouring of U is a function $f : U \to C$ so $f(x) = y$ means $x \in U$ has colour $y \in C$.

If $g \in G$ is a symmetry of U and $f : U \to C$ define

$$(gf)(u) := f(g(u))$$ so gf is a new colouring of U

Consider the # of fixed points of g in the set of C-colourings of U: i.e. $f: gf = f$ i.e. $(gf)(u) = f(u)\ \forall u \in U$

$\Rightarrow \quad f(g(u)) = f(u)\ \forall u \in U.$

$\Rightarrow \quad f(u) = f(g(u)) = f(g^2(u)) = f(g^3(u)) = \ldots$

$\Leftrightarrow$ f assigns the same colour to each member of the cycle of g

i.e. $\{g^n(u) : n \in \mathbb{Z}\}$.

$$\Rightarrow \quad \text{fix}(g) = \#\{\text{function } f \text{ from cycles of } g \text{ to } C\}$$
$$= |C|^{\#\text{cycles}(g)}$$

where a cycle of g is an orbit in U of the powers of g.

$\therefore$ # different colourings up to G equivalence $= \dfrac{1}{|G|} \displaystyle\sum_{g \in G} |C|^{\#\text{cycles}(g)}$

Ex Count the number of necklaces, with no clasp, with p (a prime) number of beads, using a different colours?

$U = \{0, 1, 2, \ldots, p-1\} = \mathbb{Z}_p$ or $\mathbb{Z}/p\mathbb{Z}$, $|U| = p$.

$G = \mathbb{Z}_p = \{0, 1, 2, \ldots, p-1\}$ acting by addition mod p.

Since x, $x + i$, $x + 2i$, $x + 3i$, $x + pi = x$, $1 \le i \le p-1$

is one cycle of length p we have $p-1$ of these and

$i = 0 \rightarrow x + 0 = x$ gives p cycles of length 1.

$$\#\text{ necklaces} = \frac{1}{p} \sum_{i \in \mathbb{Z}_p} a^{\#\text{cycles}(i)} = \frac{1}{p}\left((p-1)a^1 + a^p\right)$$

$$= a + \frac{a^p - a}{p}$$

Note: Since this is an integer p divides $a^p - a \Rightarrow a^p \equiv a \bmod p$,

which is Fermat's Little Theorem.

Sieve Method — Principle of Inclusion and Exclusion.

Given a finite set Ω of objects and a set P of properties

How many objects have no properties at all.? How many have exactly r properties? etc.

Ex How many integers $\{1, 2, 3, \ldots, N\}$ are not divisible by $\{2, 3, 5\}$.

Method converts 'at least' information into 'exact' information

If $S \subset P$ is a subset of the properties let $\boxed{N^S}$ be the number of objects in Ω that have at least the properties in S.

For $r \geq 0$ fixed let

$$N_r := \sum_{|S|=r} N^S \quad \sim \text{easier.}$$

If $x \in \Omega$ let $\boxed{P(x)}$ be the properties in P satisfied by x.

Then

$$N_r = \sum_{|S|=r} \sum_{\substack{x \in \Omega \\ S \subset P(x)}} 1$$

$$= \sum_{x \in \Omega} \left\{ \sum_{\substack{|S|=r \\ S \subset P(x)}} 1 \right\}$$

$$= \sum_{x \in \Omega} \binom{|P(x)|}{r}$$

Therefore if an object x has exactly $t = |P(x)|$ properties, it contributes $\binom{t}{r}$ to N_r. Let $e_t = \#\{x \in \Omega : |P(x)| = t\} \Rightarrow$

$$N_r = \sum_{t \geq 0} \binom{t}{r} e_t \qquad r = 0, 1, 2, \ldots \qquad \text{———} \quad (2)$$

We want to 'solve' for the e_t's as functions of the N_r's

(note $\binom{t}{r} := 0$ if $r > t$.)

By 2
$$N(x) = \sum_{r \geq 0} N_r x^r$$
$$= \sum_{r\geq 0} \sum_{t \geq 0} \binom{t}{r} e_t x^r$$
$$= \sum_{t\geq 0} e_t \sum_{r\geq 0} \binom{t}{r} x^r$$
$$= \sum_{t \geq 0} e_t (x+1)^t$$
$$= E(x+1).$$

Thus $\boxed{E(x) = N(x-1)}$ which is the sieve method!

If we know the N_r's we can read off the e_t's

Ex e_0 = # objects which have no properties
$$= E(0) = N(-1) = \sum_{t \geq 0} (-1)^t N_t$$
e_j = coeff of x^j in $\sum_{r\geq 0} N_r (x-1)^r$
$$= \sum_{r \geq 0} (-1)^{r-j} \binom{r}{j} N_r$$ = # objects with exactly j properties.

Ex Of the $n!$ permutations of $\{1, \dots, n\}$ how many have exactly r fixed pts?

A Objects = Ω = all permutations on $\{1,..,n\}$, $|\Omega| = n!$
Properties P, $|P| = n$, property i is sat. by $\tau \in \Omega$ if $\tau(i) = i$.
$1 \leq i \leq n$, i.e. τ fixes i.

B Find N^S. Let $S \subset \{1,..,n\}$. Need to count permutations τ that fix at least the elements of S. Such a τ acts freely on elements outside S i.e. $\exists$ $(n-|S|)! = N^S$.

C Calculate the N_r's :

for $r = 0, \dots, n$

$$N_r = \sum_{|S|=r} N^S = \sum_{|S|=r} (n-|S|)! = \binom{n}{r}(n-r)! = \frac{n!}{r!}$$

<u>Thus</u> $N(x) = \sum_{r=0}^{n} \frac{n!}{r!} x^r = n! \sum_{r=0}^{n} \frac{x^r}{r!}$ so

$$e_t = \text{coef of } x^t \text{ in } N(x-1)$$

& $$E(x) = \sum_t e_t x^t = n! \sum_{r=0}^{n} \frac{(x-1)^r}{r!}$$

so number of permutations with <u>no</u> fixed points is

$$e_0 = E(0) = n! \sum_{r=0}^{n} \frac{(-1)^r}{r!} \sim \frac{n!}{e} \text{ as } n \to \infty.$$

and $e_t \sim \frac{1}{e}\frac{n!}{t!}$ as $n \to \infty$

<u>Ex</u> <u>Sieving for primes</u>

(A) Let $\Omega = \{ n \in \mathbb{N} : \sqrt{N} < n \leq N \}$, $N \in \mathbb{N}$ 'large'.

Let for $2 \leq q \leq \sqrt{N}$ q a prime P_q is the property

"n is divisible by q i.e. $n = mq$ some $m \in \mathbb{N}$.

(B) $$N^S = \#\{ n : \sqrt{N} < n \leq N \text{ and } q \mid n \ \forall q \in S \}$$

If $S = \{q_1, \dots q_\ell\}$ let $q^S = q_1 \dots q_\ell$ their product. Then

$$N^S = \left\lfloor \frac{N}{q^S} \right\rfloor - \left\lfloor \frac{\sqrt{N}}{q^S} \right\rfloor, \text{ where } \lfloor x \rfloor \text{ is the integer part of } x \in \mathbb{R}.$$

(C) $$N_r = \sum_{|S|=r} N^S = \underbrace{\sum_{q_1<q_2<\dots<q_r} \left\lfloor \frac{N}{q_1 \dots q_r} \right\rfloor - \sum_{q_1<q_2<\dots<q_r} \left\lfloor \frac{\sqrt{N}}{q_1 \dots q_r} \right\rfloor}$$

<u>and so</u> $$e_0 = \text{primes in } \Omega = \sum_{r \geq 0} (-1)^r \{ \ \downarrow \ \}$$

This is a model of the <u>sieve of Eratosthenes</u>.