Enumeration Paradigms

(finding generating functions for sequences (a_n))

1) $\boxed{a_n = P(n)}$ a polynomial in n of degree $d \in \mathbb{N}$. Then

$$f(x) = \sum_{n=0}^{\infty} a_n x^n = \frac{Q(x)}{(1-x)^{d+1}}$$ where $Q(x)$ is a polynomial

Ex $\quad P(n) = b_1 n + b_0 = a_n$

$$f(x) = \sum_{n=0}^{\infty} (b_1 n + b_0) x^n \quad \text{and} \quad d = 1$$

$$= b_1 \left(\sum_{n=0}^{\infty} n x^n \right) + b_0 \sum_{n=0}^{\infty} x^n$$

But $\quad \frac{1}{1-x} = 1 + x + x^2 + \dots = \sum_{n=0}^{\infty} x^n \quad$ for $|x| < 1$

and $\quad \frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + \dots = \sum_{n=1}^{\infty} n x^{n-1}$

so $\quad \frac{x}{(1-x)^2} = x + 2x^2 + 3x^3 + \dots = \sum_{n=1}^{\infty} n x^n = \sum_{n=0}^{\infty} n x^n$

$$\therefore \quad f(x) = b_1 \frac{x}{(1-x)^2} + b_0 \frac{1}{1-x} = \frac{(b_1 - b_0)x + b_0}{(1-x)^2}$$

2) a_n satisfies a linear recurrence with constant coef c_i and a fixed number of terms.

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_d a_{n-d}, \quad d \geq 1.$$

Then $\quad f(x) = \frac{P(x)}{Q(x)} \quad$ a rational function

If there are d given coefficients we need also d starting values $a_0, a_1, a_2 \dots a_{d-1}$

Ex Fibonacci $\quad a_0 = 0, a_1 = 1 \quad a_n = a_{n-1} + a_{n-2}, \quad c_1 = 1 = c_2.$

$$\circ\circ\circ \Rightarrow \quad f(x) = \frac{x}{1-x-x^2} = \frac{P(x)}{Q(x)}.$$

<u>General derivation</u>

$$f(x) = \sum_{n=0}^{\infty} a_n x^n = \sum_{n=d}^{\infty}\left(\sum_{i=1}^{d} c_i \cdot a_{n-i}\right) + \underbrace{a_0 + a_1 x + \dots + a_{d-1} x^{d-1}}_{S(x)}$$

$$= \sum_{i=1}^{d} \sum_{n=d}^{\infty} c_i \cdot a_{n-i} x^{n-i} \cdot x^i + S(x)$$

$$= \sum_{i=1}^{d} c_i x^i \left(\sum_{n=d}^{\infty} a_{n-i} x^{n-i}\right) + S(x)$$

$m = n - i \Rightarrow$
$$= \sum_{i=1}^{d} c_i x^i \sum_{m=d-i}^{\infty} a_m x^m + S(x)$$

ut $\displaystyle\sum_{m=0}^{\infty} a_m x^m = \sum_{m=0}^{d-i-1} a_m x^m + \sum_{m=d-i} a_m x^m$ so therefore

$$f(x) = \sum_{i=1}^{d} c_i x^i \left[f(x) - \sum_{m=0}^{d-i-1} a_m x^m \right] + S(x)$$

$$= \sum_{i=1}^{d} R_i(x) \left[f(x) - T_i(x) \right] + S(x)$$

$$= f(x) \sum_{i=1}^{d} R_i(x) - \sum_{i=1}^{d} R_i(x) T_i(x) + S(x)$$

$$\therefore \quad f(x) = \frac{S(x) - \sum_{i=1}^{d} R_i(x) T_i(x)}{1 - \sum_{i=1}^{d} R_i(x)} = \frac{P(x)}{Q(x)} \quad //$$

<u>Note</u> In practice the general result is <u>not</u> used, but we erive $f(x)$ directly, going through these same steps, see Fibonacci like example.

<u>Ex</u> (not in the class)

$$\boxed{a_{n+1} = 2a_n + 1 \,, \quad n \geq 0 \,, \quad a_0 = 0} \qquad (a_n) = (0, 1, 3, 7, 15, 31, \dots)$$

<u>Claim</u> $a_n = 2^n - 1 \quad \forall\, n \geq 0$. Prove by induction. OR.

Let $\displaystyle f(x) = \sum_{n=0}^{\infty} a_n x^n$.

Then $\displaystyle\sum_{n=0}^{\infty} a_{n+1}x^n = \frac{(a_0 + a_1 x + a_2 x^2 + \dots) - a_0}{x} = \frac{f(x)}{x}$

and $\therefore$ $\displaystyle\frac{f(x)}{x} = \sum_{n=0}^{\infty} 2a_n x^n = 2\sum_{n=0}^{\infty} a_n x^n = 2f(x) + \frac{1}{1-x}$

$\displaystyle + \sum_{n=0}^{\infty} x^n \qquad + \frac{1}{1-x}$

$\displaystyle\therefore \quad \frac{f(x)}{x} = 2f(x) + \frac{1}{1-x} \Rightarrow \boxed{f(x) = \frac{x}{(1-x)(1-2x)}}$

So we have our "clothesline". Now we need to extract a_n.

Use partial fractions $\displaystyle\frac{x}{(1-x)(1-2x)} = x\left(\frac{2}{1-2x} - \frac{1}{1-x}\right)$

$$= 2x(1 + 2x + 2^2x^2 + \dots) - x(1 + x + x^2 + \dots) \quad \Big\} \text{ Geometric series}$$

$$= (2-1)x + (2^2-1)x^2 + \dots + (2^n-1)x^n \dots$$

i.e. $a_0 = 0$ and $\boxed{a_n = 2^n - 1}$ as claimed.

Ex $\boxed{a_{n+1} = 2a_n + n, \quad n \geq 0, \quad a_0 = 1}$ $\qquad (a_n) = (1, 2, 5, 12, \dots)$.

$\displaystyle f(x) = \sum_{n=0}^{\infty} a_n x^n$ so $\displaystyle (a_1 + a_2 x + a_3 x^2 + \dots a_{n+1}x^n + \dots) = \frac{f(x) - a_0}{x}$

$$\sum_{n=0}^{\infty} a_{n+1}x^n = \sum_{n=0}^{\infty} (2a_n + n)x^n = 2f(x) + \sum_{n=0}^{\infty} nx^n$$

and, by differentiating the geometric series $\displaystyle\sum_{n=0}^{\infty} nx^n = \frac{x}{(1-x)^2}$

$$\therefore \quad \frac{f(x)-1}{x} = 2f(x) + \frac{x}{(1-x)^2} \Rightarrow \boxed{f(x) = \frac{1-2x+2x^2}{(1-x)^2(1-2x)}}$$

The partial fractions have the form $\displaystyle\frac{A}{(1-x)^2} + \frac{B}{1-x} + \frac{C}{1-2x} = \frac{-1}{(1-x)^2} + \frac{2}{1-2x}$

Diff the geometric series gives the coef of x^n in $\displaystyle\frac{1}{(1-x)^2}$ as $(n+1)$

Using the geometric series $\displaystyle\frac{1}{1-2x}$ gives coef of x^n in $\displaystyle\frac{2}{1-2x}$ as 2^{n+1}.

Hence $\boxed{a_n = 2^{n+1} - (n+1)}$ $\quad$ Check $a_0 = 2^1 - 1 = 1$ ✓

$a_1 = 2^2 - 2 = 2$ ✓

Method using the Characteristic equation

Given $\quad a_n = c_1 a_{n-1} + \dots + c_d a_{n-d} \qquad n \geq d$

Try $\quad a_n = r^n$, r unknown.

Then if the a_n is a solution we must have

$$r^n = c_1 r^{n-1} + \dots + c_{d-1} r^{n-d+1} + c_d r^{n-d}$$

i.e. $\qquad 0 = P(r) = r^d - c_1 r^{d-1} - \dots - c_d \;, \quad \partial P = d.$

If we can solve this polynomial and obtain distinct roots $\{r_1, \dots, r_d\}$

we can solve the linear system

$$a_n = \sum_{d=1}^{d} \alpha_j r_j^n$$

$$a_0 = \sum_{d=1}^{d} \alpha_j r_d^0$$

$$a_1 = \sum_{d=1}^{d} \alpha_j r_d^1$$

$$\vdots$$

$$a_{d-1} = \sum_{d=1}^{d} \alpha_j r_j^{d-1}$$

$$\overset{d=3}{\Longrightarrow} \begin{pmatrix} 1 & 1 & 1 \\ r_1 & r_2 & r_3 \\ r_1^2 & r_2^2 & r_3^2 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix}$$

Ex $\quad \boxed{a_n = a_{n-1} + a_{n-2} \,, \quad a_0 = 0, \; a_1 = 1}$ Fibonacci

$a_n = r^n$: $\quad r^n = r^{n-1} + r^{n-2} \qquad n \geq 2$

$\div r^{n-2}$: $\quad r^2 = r + 1 \quad \Rightarrow \quad r^2 - r - 1 = 0$

Solve $\qquad r = \dfrac{1 \pm \sqrt{5}}{2} \quad \Rightarrow \quad r_1 = \dfrac{1+\sqrt{5}}{2}, \; r_2 = \dfrac{1-\sqrt{5}}{2}$

Find the solution satisfying the initial conditions

$$a_n = \alpha_1 r_1^n + \alpha_2 r_2^n \quad \Rightarrow \quad \left\{ \begin{aligned} a_0 = 0 &= \alpha_1(1) + \alpha_2(1) \\ a_1 = 1 &= \alpha_1\left(\tfrac{1+\sqrt{5}}{2}\right)^1 + \alpha_2\left(\tfrac{1\pm\sqrt{5}}{2}\right)^1 \\ &= \alpha_1\left(\tfrac{1+\sqrt{5}}{2} - \tfrac{1-\sqrt{5}}{2}\right) \\ &= \alpha_1(\sqrt{5}) \end{aligned} \right.$$

$$\Longrightarrow \quad a_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]$$

Check $\quad a_0 = \dfrac{1}{\sqrt{5}}\left[1 - 1\right] = 0$

$\qquad\quad a_1 = \dfrac{1}{\sqrt{5}}\left[\dfrac{1+\sqrt{5}}{2} - \left(\dfrac{1-\sqrt{5}}{2}\right)\right] = 1$