Ex (3) (well formed bracketings)

If L_1 and L_2 are legal bracketings then $[L_1]L_2$ is also legal.

Ex $L_1 = [\,][\,]$, $L_2 = [[\,]]$ $\longrightarrow$ $\underbrace{[[\,][\,]]}_{[L_1]}\underbrace{[[\,]]}_{L_2}$

Let A_n be the set of bracketings with n pairs. If it splits and L_1 has k pairs then L_2 has $n-k-1$ pairs

$$\therefore \quad A_n = \bigsqcup_{k=0}^{n-1} A_k \times A_{n-k-1}$$

So $$a_n = \sum_{k=0}^{n-1} a_k\, a_{n-k-1},$$ a non-linear recurrence.

– the decomposition $[L_1]L_2$ is unique – we must take the first matching RH bracket.

– the union is disjoint $\because$ an element of A_k is never equal to an element of $A_{k'}$ for $k \neq k'$.

7 Generating Functions

Wilf – "A generating function is a clothesline in which we hang up a sequence of numbers for display"

– Given a set of numbers (not necessarily in $\mathbb{N}$ or in $\mathbb{R}$ even) (a_n), a generating function contains all of the information in all of the numbers.

– it can be used to obtain each of the numbers.

Ex 1 $$e^x = 1 + x + \frac{x^2}{2!} + \dots + \frac{x^n}{n!} + \dots = \sum_{n=0}^{\infty} a_n x^n$$

is a generating function for $a_n = \frac{1}{n!}$

Ex 2 $$\left(\frac{1}{1-x}\right)^2 = 1 + 2x + 3x^2 + 4x^3 + \dots$$ generates $\mathbb{N}$.

Ex 3 $\quad \dfrac{x}{e^x - 1} = B_0 + B_1 x + B_2 x^2 + \ldots$ generates the Bernoulli numbers.

The terms in the sequence are regarded as coefficients of a power series in x : $\quad f(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n + \ldots$

$$a_n = \frac{f^{(n)}(0)}{n!}$$ if the function is real analytic.

Ex 4 $\quad a_0 = a_1 = 1, \quad a_n = a_{n-1} + a_{n-2} \quad n \geq 2$ then

$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$= a_0 + a_1 x + \sum_{n=2}^{\infty} a_n x^n$$

$$= 1 + x + \sum_{n=2}^{\infty} (a_{n-1} + a_{n-2}) x^n$$

$$= 1 + x + \sum_{n=2}^{\infty} a_{n-1} x^n + \sum_{n=2}^{\infty} a_{n-2} x^n$$

$$= 1 + x + x \sum_{n=2}^{\infty} a_{n-1} x^{n-1} + x^2 \sum_{n=2}^{\infty} a_{n-2} x^{n-2}$$

$$\Rightarrow f(x) = 1 + x + x(f(x) - 1) + x^2(f(x))$$

$$= 1 + (x + x^2) f(x)$$

Hence $\boxed{f(x) = \dfrac{1}{1 - x - x^2}}$, a nice rational function which thus embodies all of the information "in" the Fibonacci like numbers.

Do we need to be concerned with the convergence of these series? – often no. They are regarded as 'formal power series'. E.g.

$\sum_{n=0}^{\infty} n!\, x^n$ as a Taylor series converges only at $x = 0$,

$= \sum_{n=0}^{\infty} a_n x^n$ and $a_n = n!$ is still implied.

Weight Enumeration

If $f(x) = \sum_{n=0}^{\infty} a_n x^n$, we regard this as a "weight enumerator" of a family of combinatorial sets (A_n) with $a_n =$ "weight"(A_n), usually $a_n = |A_n|$.

Ex Let q_i be the number of people in a population with age i. $1 \le i \le 200$. We want $f(x) = \sum_{i=0}^{200} q_i x^i$.

1st method In turn count the number of people of age i, $1 \le i \le 200$. So we need 201 questions and answers.

2nd method – good if the population has less than 200 members – ask each person their age and say the weight of a person of age i is x^i so

$$f(x) = \sum_{p \in \text{persons}} x^{\text{age}(p)}$$

Ex

person:	Jack	Ellen	Mary	Tom	Ben	Jan
age:	18	19	21	19	18	18

$$f(x) = x^{18} + x^{19} + x^{21} + x^{19} + x^{18} + x^{18}$$

$$= 3x^{18} + 2x^{19} + x^{21}$$

so there are 3 of age 18, 2 of age 19 and 1 of age 21 & no others.
— 6 steps instead of 200.

Total no. of individuals is $3 + 2 + 1 = f(1)$

Total age is $3 \times 18 + 2 \times 19 + 1 \times 21 = f'(1)$

Average age is $\mu = \dfrac{f'(1)}{f(1)}$.

and Variance $\sigma^2 = \sum_i (a_i - \mu)^2 / (\text{total no. of individuals})$

$a_i =$ age of person i $\quad = \dots \dfrac{f''(1)}{f(1)} + \mu - \mu^2$.

General Scheme

Given a set A and an attribute of elements of A $\alpha : A \to \mathbb{N}' = \mathbb{N} \cup \{0\}$, then the weight enumerator of A with respect to α is defined by

$$f(x) := \sum_{a \in A} x^{\alpha(a)} =: |A|_x$$

$$= \sum_{n=1}^{\infty} a_n x^n$$

where $a_n = \#\{ a \in A : \alpha(a) = n \}$. So once we have an explicit $f(x)$ we can compute the a_n's.

Properties of weighted counting

① $$|A \uplus B|_x = |A|_x + |B|_x$$

$$\text{LHS} = \sum_{a \in A \uplus B} x^{\alpha(a)} = \sum_{a \in A} x^{\alpha(a)} + \sum_{a \in B} x^{\alpha(a)} = \text{RHS}$$

② $$|A \times B|_x = |A|_x \cdot |B|_x$$

Let α be the attribute of A, β . . . . B, and let γ on $A \times B$ be defined by $\gamma(a,b) = \alpha(a) + \beta(b)$.

then $\text{LHS} = |A \times B|_x = \sum_{(a,b) \in A \times B} x^{\gamma(a,b)}$

$$= \sum_{(a,b) \in A \times B} x^{\alpha(a) + \beta(b)}$$

$$= \sum_{(a,b) \in A \times B} x^{\alpha(a)} \cdot x^{\beta(b)}$$

$$= \sum_{a \in A} \sum_{b \in B} x^{\alpha(a)} \cdot x^{\beta(b)}$$

$$= \left(\sum_{a \in A} x^{\alpha(a)} \right) \cdot \left(\sum_{b \in B} x^{\beta(b)} \right)$$

$$= \text{RHS}.$$

Ex Let A be the set of all finite sequences of 1s and 2s and let the ~~weight~~ attribute of a sequence $a \in A$, $\alpha(a)$ be the sum of its entries, so e.g. $\alpha(121) = 4$ so the weight of the sequence is $x^{\alpha(a)} = x^4$. $\text{Weight}(a_1 .. a_n) = x^{a_1+a_2+\cdots+a_n}$.

Then
$$A = \phi \sqcup 1A \sqcup 2A$$
where $1A$ represents the subset of sequences commencing with 1, etc.

then
$$|A|_x = 1 + x|A|_x + x^2|A|_x \Rightarrow$$
$$|A|_x = \frac{1}{1-x-x^2} \quad (\text{Fibonacci}).$$

Ex Let $\mathcal{L}$ be the set of legal bracketings. Then the empty bracketing has weight $x^0 = 1$. Otherwise we can write a bracketing as $L = [L_1]L_2$ where L_1, L_2 are legal. Let the weight of $L := x^n$ where n is the no. of matched pairs. e.g. $wt([[\,][\,]]) = x^3$
$$\mathcal{L} = \{\phi\} \sqcup ([\mathcal{L}] \times \mathcal{L})$$
$$\Rightarrow \quad |\mathcal{L}|_x = 1 + x|\mathcal{L}|_x \cdot |\mathcal{L}|_x = 1 + x|\mathcal{L}|_x^2$$
$$\Rightarrow \quad |\mathcal{L}|_x = \frac{1-\sqrt{1-4x}}{2x} = \sum_{n=0}^{\infty} a_n x^n$$

Binomial theorem for fractional indices $\alpha \in \mathbb{R}$ and $|x| < 1$
$$(1+x)^\alpha = 1 + \alpha x + \frac{(\alpha)(\alpha-1)}{1\cdot 2}x^2 + \cdots + \binom{\alpha}{j}x^j + \cdots \quad \binom{\alpha}{j} := \frac{\alpha(\alpha-1)..(\alpha-j+1)}{j!}$$
Replace $x \to -4x$, $\alpha \to \frac{1}{2}$ to expand $(1-4x)^{1/2}$ to derive
$$a_n = \frac{(2n)!}{(n+1)!\,n!} \quad (3).$$

Finding solutions : Solving recurrence relations

$$\text{From } (a_n) \longleftrightarrow f(x) = \sum_0^\infty a_n x^n$$

Ex 1 $a_0 = 3,\ a_1 = 6,\ a_{n+2} = 3a_{n+1} - 2a_n,\ n \geq 0.$

so $a_2 = 3a_1 - 2a_0 = 12$ etc.

then $a_{n+2} - 3a_{n+1} + 2a_n = 0,\ n \geq 0.$

$$\Rightarrow \sum_{n=0}^\infty (a_{n+2} - 3a_{n+1} + 2a_n)x^n = 0$$

$$\Rightarrow \sum_{n=0}^\infty a_{n+2}x^n - 3\sum_{n=0}^\infty a_{n+1}x^n + 2\sum_{n=0}^\infty a_n x^n = 0$$

$$x(x^2) \Rightarrow \sum_{n=0}^\infty a_{n+2}x^{n+2} - 3x\sum_{n=0}^\infty a_{n+1}x^{n+1} + 2x^2\sum_{n=0}^\infty a_n x^n = 0$$

$$\Rightarrow (f(x) - a_0 - a_1 x) - 3x(f(x) - a_0) + 2x^2 f(x) = 0$$

so $f(x)(1 - 3x + 2x^2) + (3x - 3) = 0$

$$\Rightarrow f(x) = \frac{3 - 3x}{1 - 3x + 2x^2} = \frac{3(1-x)}{(1-x)(1-2x)}$$

$$= \frac{3}{1-2x} = 3(1 + 2x + 4x^2 + 8x^3 + \dots)$$

using $\frac{1}{1-r} = 1 + r + r^2 + \dots,\ r = 2x,\ r^n = 2^n x^n.$

so $\boxed{a_n = 3.2^n}\ \forall n \geq 0$

Ex 2 (Fibonacci) $a_0 = 0,\ a_1 = 1\ \ a_n = a_{n-1} + a_{n-2},\ n \geq 2.$

.... we get $f(x) = \dfrac{x}{1 - x - x^2}$. Let

$\gamma = \frac{1+\sqrt{5}}{2}$ be the golden ratio and

$\delta = \frac{1-\sqrt{5}}{2}$ its conjugate $1/\gamma$ so $(1 - x - x^2) = (1-\gamma x)(1-\delta x)$

and ... $f(x) = \dfrac{1}{\sqrt{5}}\left(\dfrac{1}{1-\gamma x} - \dfrac{1}{1-\delta x}\right) \Rightarrow a_n = \dfrac{\gamma^n - \delta^n}{\sqrt{5}}.$